A metric Lie algebra is a finite-dimensional real Lie algebra equipped with an invariant nondegenerate symmetric bilinear form. An example of a metric Lie algebra is the Lie algebra of the transvection group of a pseudo-Riemannian symmetric space. The paper contains a systematic study of the structure of metric Lie algebras. A metric Lie algebra $g$ is indecomposable if it does not contain a proper ideal on which the bilinear form is nondegenerate, and in this case $g$ is either simple or does not contain a simple ideal.

Let $l$ be a Lie algebra and $(\rho, a)$ be an orthogonal $l$-module. A quadratic extension of $l$ by $a$ is a quadruple $(g, i, i, p)$, where $g$ is a metric Lie algebra, $i$ is an isotropic ideal in $g$, and $i : a \to g/i$ and $p : g/i \to l$ are Lie algebra homomorphisms such that $0 \to a \to g/i \to l \to 0$ is an exact sequence, $i(\rho(L)A) = [\tilde{L}, i(a)] \in i(a)$ holds for all $A \in a$ and $\tilde{L} \in g/i$ with $\rho(\tilde{L}) = L$, $i(a) = i^2/i$ and $i : a \to i^2/i$ is an isometry. Such a quadratic extension does not contain a simple ideal. The authors show that any metric Lie algebra $g$ without simple ideals has the structure of a balanced quadratic extension. Here, balanced means that $i$ coincides with a canonical isotropic ideal $i(g)$ in $g$. For a balanced quadratic extension the representation $\rho$ of $l$ on $a$ is semisimple.

The balanced quadratic extensions can be identified with elements in certain quadratic cohomology sets $H^2_Q(l,a)$. The investigation of $H^2_Q(l,a)$ is the main technical tool in the paper. The authors show that there is a one-to-one correspondence between the set of balanced quadratic extensions and the set $H^2_Q(l,a) \circ/G_{l,a}$ of certain admissible cohomology classes in $H^2_Q(l,a)$. This leads to the characterization of the set of isomorphism classes of nonsimple indecomposable metric Lie algebras as the union $\Pi_{l,a}H^2_Q(l,a) \circ/G_{l,a}$ over a set of representatives of isomorphism classes of pairs $(l,a)$ consisting of real finite-dimensional Lie algebras $l$ and a semisimple orthogonal $l$-module $a$, where $G_{l,a}$ is the automorphism group of the pair $(l,a)$ and the orbit space $H^2_Q(l,a) \circ/G_{l,a}$ is with respect to the induced action of $G_{l,a}$ on $H^2_Q(l,a) \circ$. This characterization of nonsimple indecomposable metric Lie algebras by certain equivalence classes of cohomology sets indicates a classification scheme for such Lie algebras. The authors use this scheme to obtain an explicit classification of all nonsimple indecomposable metric Lie algebras of index 3.

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