For most large underdetermined systems of linear equations the minimal $\ell_1$-norm solution is also the sparsest solution. (English) Zbl 1113.15004 Commun. Pure Appl. Math. 59, No. 6, 797-829 (2006).

The author studies sparse solutions to underdetermined linear systems $y = \Phi x$ where $y \in \mathbb{R}^n$ is a given vector and $\Phi$ is a given $n \times m$ matrix with $n < m \leq \tau n$. In the last years, a large body of research has focused on the use of overcomplete signal representations. If we have more atoms $\phi_i \in \Phi$ than observation dimensions, $m > n$, then there are many possible representations $\Phi x = y$ for a given $\Phi$ and $y$. The sparse representation problem is then to find the representation $x$ with the fewest possible non-zero components,

$$\min \|x\|_0 \text{ subject to } \Phi x = y,$$

where $\| \|_0$ is the $l_0$ norm of a vector $x$, i.e., the number of its non-zero coefficients. This is well known to be a NP-hard problem [cf. B. K. Natarajan, SIAM J. Comput. 24, No. 2, 227–234 (1995; Zbl 0827.68054)].

In the signal processing community, S. S. Chen, D. L. Donoho and M. A. Saunders [SIAM J. Sci. Comput. 20, No. 1, 33–61 (1999; Zbl 0919.94002)] proposed to approximate $(P_0)$ with the ‘relaxed’ $l_1$ problem

$$\min \|x\|_1 \text{ subject to } \Phi x = y,$$

where $\| \|_1 = \sum_{i=1}^m |x_i|$ is the $l_1$ norm of a vector $x$. Problem $(P_1)$, which they called basis pursuit (BP), can be formulated as a linear programming (LP) problem, which can be solved using well known optimization methods such as the simplex method or interior point methods [cf. M. H. Wright, Bull. Am. Math. Soc., New Ser. 42, No. 1, 39–56 (2005; Zbl 1114.90153)]. Surprisingly, when the answer to $(P_0)$ is sparse, it can be the same as the answer to $(P_1)$.

The equivalence breakdown point of a matrix $\Phi$, $\text{EBP}(\Phi)$, is the maximal number $N$ such that, for every $x_0$ with fewer than $N$ nonzeros, the corresponding vector $y = \Phi x_0$ generates a linear system $y = \Phi x$ for which problems $(P_0)$ and $(P_1)$ have identical unique solutions, both equal to $x_0$.

The main result of the paper is: For each $\tau > 1$, there exists a constant $\rho^*(\tau) > 0$ so that for every sequence $(m_n)$ with $m_n \leq \tau n$,

$$\text{Prob}\{n^{-1}\text{EBP}(\Phi_{n,m_n}) \geq \rho^*(\tau)\} \to 1, \quad n \to \infty.$$

In other words, for the overwhelming majority of $n \times m$ matrices $\Phi$, the sparsest possible solution and the minimal $l_1$-solution of $y = \Phi x$ coincide whenever a solution with at most $\rho^* n$ nonzeros exists.


The paper finishes showing as two standard approaches, simple thresholding and greedy subset selection algorithms, in contrast to the $l_1$-minimization, which are poorly performed in the setting of the paper.

Reviewer: Manuel Ladra Gonzalez (Santiago)
MSC:
15A06 Linear equations (linear algebraic aspects)
65F20 Numerical solutions to overdetermined systems, pseudoinverses
65F50 Computational methods for sparse matrices
60D05 Geometric probability and stochastic geometry
41A65 Abstract approximation theory (approximation in normed linear spaces and other abstract spaces)
52A41 Convex functions and convex programs in convex geometry
90C20 Quadratic programming

Keywords:
solution of underdetermined linear systems; overcomplete representations; minimum $l_1$-decomposition; sparse representation problem; almost-Euclidean sections of Banach spaces; eigenvalues of random matrices; sign-embeddings in Banach spaces; greedy algorithms; matching pursuit; basis pursuit

Software:
lars

Full Text: DOI

References:

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