One of the important consequences of the Hahn-Banach theorem is the so-called “extension property of $\ell_\infty$.” It states that, given a normed space $X$ and a subspace $Y \subset X$, every linear operator $S : Y \to \ell_\infty$ can be extended to an operator $T : X \to \ell_\infty$ having the same norm as $S$. This theorem is used in the proofs of many results in Banach space theory and related fields. In particular, it was one of the ingredients of the following result on covering numbers, obtained recently in A. E. Litvak, A. Pajor, and N. Tomczak-Jaegermann [J. Funct. Anal. 231, No. 2, 438–457 (2006; Zbl 1092.52004)]: Let $0 < a < r < A$ and $1 \leq k < n$. Let $K, L \subset \mathbb{R}^n$ be symmetric convex bodies, and let $K \subset aL$. Let $E \subset \mathbb{R}^n$ be a $k$-codimensional subspace such that $K \cap E \subset aL$. Then

$$N(K, 2rL) \leq 2^k \left( \frac{A + r}{r - a} \right)^k,$$

where $N(K, L)$ denotes the covering number. In some sense, the last result is a (weak) version of the extension theorem for entropy: if we control the norm of the identity operator (= half diameter of the unit ball) in a subspace, then we control entropy in the entire space. Note that if $K \cap E \subset aL$, then trivially $N(K \cap E, aL) \leq 1$. However, why should the diameter play such a crucial role? Can one achieve a similar control of entropy in the entire space from the knowledge of entropy (rather than the diameter) in a subspace? Intuition does not support such a hope. However, quite surprisingly, this is possible.

The authors prove a strong version of an extension theorem for entropy: if we control entropy in a subspace, then we control entropy in the entire space. The main result of this paper is the “entropy extension-lifting theorem.” It consists of two inequalities for entropies. The first inequality relates the entropy of $K$ and $L$ to the entropy of sections of small codimension and can be called an “entropy extension theorem”, while the second inequality assumes information on entropies of projections of small corank and can be called an “entropy lifting theorem”. The authors also provide a version of the inverse statement and discuss the nonsymmetric case.

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**MSC:**

- 52A40  Inequalities and extremum problems involving convexity in convex geometry
- 52A22  Random convex sets and integral geometry (aspects of convex geometry)
- 37B40  Topological entropy
- 47B06  Riesz operators; eigenvalue distributions; approximation numbers, $s$-numbers, Kolmogorov numbers, entropy numbers, etc. of operators
- 54C70  Entropy in general topology
- 52C17  Packing and covering in $n$ dimensions (aspects of discrete geometry)
- 46B07  Local theory of Banach spaces

**Keywords:**

- metric entropy
- Hahn-Banach theorem
- entropy extension-lifting theorem
- entropy decomposition
- covering numbers
- nonsymmetric body

**Full Text:** DOI

**References:**


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