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Algebraic theory of locally nilpotent derivations. (English) Zbl 1121.13002

Encyclopaedia of Mathematical Sciences 136. Invariant Theory and Algebraic Transformation Groups 7. Berlin: Springer (ISBN 3-540-29521-6/hbk). 260 p. (2006).

Let A be a commutative ring and M an A-module. A derivation D is an additive map $D : A \to M$ satisfying the Leibniz rule. That is, $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$ for all $a, b \in A$. A derivation $D : A \to A$ is called locally nilpotent (abbreviated LND) if for any $a \in A$, there exists an n such that $D^n(a) = 0$. A typical example is when $A = R[x]$, polynomial ring in one variable X over R and $D = \frac{d}{dX}$, the X-derivative. Loosely speaking, the study of LNDs have been to see whether given one such, how closely does it resemble the typical one as above.

In the volume under review, the author gives a detailed description of the subject covering all the important results (and then some more). The study of locally nilpotent derivations as an independent field began less than fifty years ago, but it already has an impressive array of results which clarify and generalize earlier isolated classical results. A detailed description of what the book contains seem pointless and thus one shall just mention some of its salient features.

The subject is closely related to Hilbert's Fourteenth problem and there are several classical results in this direction dealt with in this book. Some of the earliest are due to M. Nagata (and fittingly, Professor Nagata's picture graces the cover of the book). Locally nilpotent derivations are closely related to vector group (the additive group $\mathbb{G}_a^n$) actions and that justifies the inclusion of the book in the series devoted to Invariant Theory.

One of the most important part of the study is LNDs of polynomial rings over a field (of characteristic zero). There is not much to say in the case of one variable, since they are the obvious ones. The case of two variables is completely understood, albeit non-trivial. The case of three variables is still not completely understood, but the book gives an excellent survey of the state of the art. The book also describes the more recent tools to study LNDs, like the Makar-Limanov invariant and Derksen invariant.

Finally, the book has a wealth of examples and the Epilogue details some important open problems in the area. The book is more or less self-contained, especially if you have had a reasonable course in commutative algebra. It studiously avoids the heavier machinery of geometric arguments and thus is accessible to less advanced graduate students. It is a valuable addition to the literature and am sure would be very helpful to the interested student and researcher alike.

Reviewer: N. Mohan Kumar (St. Louis)

MSC:

13-02	Research exposition (monographs, survey articles) pertaining to commutative algebra
13N10	Commutative rings of differential operators and their modules
14R20	Group actions on affine varieties
14R10	Affine spaces (automorphisms, embeddings, exotic structures, cancellation problem)

Keywords:

locally nilpotent derivations; additive group actions; Hilbert's Fourteenth Problem