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Clean rings of continuous functions. (English) Zbl 1123.54003

W. K. Nicholson introduced the concept of clean rings in [Trans. Am. Math. Soc. 229, 269–278 (1977; Zbl 0352.16006)]; these are rings in which every element is the sum of an idempotent and a unit. In the paper [Acta Math. Hungar. 94, 53–58 (2002; Zbl 0996.54023)], F. Azarpanah has shown that $C(X)$ is clean if and only if $X$ is strongly zero-dimensional, where $C(X)$ is the ring of all real-valued continuous functions on a completely regular Hausdorff space $X$. In the paper under review, the authors show in Theorem 3.7 that if $A$ is a dense clean subring of $\mathbb{R}$ which is not a field, but $1 \in A$ and if $X$ is a zero dimensional completely regular Hausdorff space, then $C(X, A)$ is a clean ring if and only if $X$ is a $P$-space. Here $C(X, A)$ is the ring of all $A$-valued continuous functions on $X$.

But the present paper gives several more important and interesting results. In particular, the authors give detailed descriptions of the prime and maximal ideal spectra, $\text{Spec}(R)$ and $\text{Max}(R)$ where $R = C(X, A)$ or $C^*(X, A) = \{f \in C(X, A) : f \text{ is bounded}\}$. In this regard, Corollary 4.3 is a very interesting and detailed result for an infinite $P$-space $X$. In order to complete the description of the spectra, the authors introduce the concepts of $c$-ideal and $Z_\beta$-ideal in $C(X, A)$ and $C^*(X, A)$, respectively. For $c$-ideals, Corollary 4.10 is an interesting result. Concerning $c$-ideals and $Z_\beta$-ideals, the authors state the following conjecture: Let $1 \in A$ be a dense local subring of $\mathbb{R}$ that is not a field and let $X$ be an infinite $P$-space. Then a maximal ideal of $C^*(X, A)$ is either a $c$-ideal or a $Z_\beta$-ideal. By assuming that the conjecture holds, it is shown that $\text{Max}(C^*(X, A))$ is homeomorphic to $\beta X \coprod (\beta X \setminus \upsilon X)$.

The paper is well written with well-organized sections and clearly presented proofs.

Reviewer: Subiman Kundu (New Delhi)

MSC:
54C30 Real-valued functions in general topology
54C40 Algebraic properties of function spaces in general topology
54G10 $P$-spaces
06F25 Ordered rings, algebras, modules
13A99 General commutative ring theory

Keywords:
Rings of continuous functions; $P$-space; clean ring; spectra; $c$-ideal; $Z_\beta$-ideal

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