Several decades ago, André Weil discovered that the classical explicit formulas of prime number theory can be generalized to the case of Artin-Hecke $L$-series over an arbitrary global field. Weil’s generalized formulas connect arithmetic expressions, namely certain character sums, and analytic expressions, basically sums over the zeros of an Artin-Hecke $L$-series, in a fundamental way. Moreover, Weil’s approach led to a statement on the positivity of a certain distribution, which is equivalent to the simultaneous validity of the Riemann hypothesis for Artin-Hecke $L$-series and the famous Artin conjecture [cf.: A. Weil, Izv. Akad. Nauk SSSR Ser. Mat. 36, 3–18 (1972; Zbl 0245.12010)].

More recently, A. Connes gave an interpretation of Weil’s generalized explicit formulas as a geometric Lefschetz trace formula over the noncommutative space of adèle classes [Sel. Math., New Ser. 5, No. 1, 29–106 (1999; Zbl 0945.11015)], thereby making transparent a striking interplay between number theory and noncommutative geometry.

The problem of extending these results to $L$-functions of algebraic varieties seems to require an even more general theory that combines noncommutative spaces and algebraic motives. On the other hand, examples of an intriguing interaction between noncommutative geometry and the theory of motives have already been encountered in recent years, for instance in the context of perturbative renormalization in quantum field theory [cf.: A. Connes and M. Marcolli, in: Frontiers in Number Theory, Physics and Geometry, Vol. II, Springer-Verlag, 617–713 (2006; Zbl 1200.81113)].

In the treatise under review, the authors provide a first systematic account of such a common framework for noncommutative geometry and motives, mainly in order to obtain a cohomological interpretation of the special realization of zeros of $L$-functions.

After a comprehensive introduction to their work (Section 1), the authors first discuss the problem of properly defining morphisms for noncommutative spaces in Section 2. In this context, it is explained that there are, in fact, at least two well-developed constructions in noncommutative geometry that allow to define morphisms as correspondences reflecting the phenomenon of Morita equivalence, namely G. G. Kasparov’s $KK$-theory, on the one hand, and the theory of modules over the cyclic category, including cyclic cohomology, on the other. Section 3 is to show how certain categories of motives can be embedded faithfully into categories of noncommutative spaces, with a special emphasis put on the category of Artin motives and its relations to a particular noncommutative space linked to adèle classes, the so-called Bost-Connes system. The authors’ construction leads to a new category, the category of endomotives, whose objects are obtained from projective systems of Artin motives with actions of semigroups of endomorphisms.

In Section 4, the authors describe a very general cohomological procedure which associates to certain noncommutative spaces of type $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital involutive algebra over $\mathbb{C}$ and $\varphi$ is a so-called “state” on $\mathcal{A}$, a representation of the multiplicative group $\mathbb{R}_+^*$. In the particular case of the Bost-Connes system, it turns out that the spectrum of this representation is precisely the set of nontrivial zeros of Hecke $L$-functions with “Grössencharakter” associated with this special set-up. Also, it is explained to what extent the action of the scaling group $\mathbb{R}_+^*$ on the cyclic homology is analogous the action of the Frobenius on the $\ell$-adic cohomology in characteristic zero, and how this construction is related to the physical aspects (thermodynamics) of noncommutative spaces.

More precisely, it is shown that the action of the scaling group is obtained through the thermodynamics of the quantum statistical system associated to an endomotive as defined in Section 2. In Section 5, the cohomological construction of the previous section is discussed in the context of general global fields and their Hecke $L$-functions. The authors mainly give a summary of their main results in this direction, the details of which will be provided in their forthcoming paper “The Weil proof and the geometry of the adèle class space” [arXiv:math/0703392]. Anyway, these results are to lay the foundations of a geometric framework in which one can begin to transpose Weil’s particular approach to the Riemann hypothesis (via a special Riemann-Hilbert correspondence) in the case of positive characteristics to the case of general number fields, thereby obtaining the desired useful cohomological interpretation of the zeros of
the Riemann zeta function.

Section 6 turns from the special zero-dimensional case of Artin motives to higher-dimensional cases, including motives of abelian varieties and Shimura varieties. In this context, the authors discuss the compatibility of morphisms of noncommutative spaces given by correspondences in the higher-dimensional setting, mainly by comparing correspondences defined by algebraic cycles with P. Baum’s topological correspondences [cf.: A. Connes and G. Skandalis, Publ. Res. Inst. Math. Sci. 20, 1139–1183 (1984; Zbl 0575.58030)], and by reformulating the case of algebraic cycles as a particular case of the KK correspondence established by A. Connes and G. Skandalis in 1984.

The final Section 7 is devoted to $L$-functions of smooth projective varieties over a number field $K$. The authors consider the archimedean local factors of the Hasse-Weil $L$-function $L(H^m(X, C), z)$ of such a variety $X$ and establish a Lefschetz trace formula for those, where their approach is based on A. Connes’s earlier geometric interpretation of A. Weil’s generalization of the classical explicit formulas in prime number theory.

All together, this highly stimulating article provides a wealth of new ideas and insights concerning the interaction of number theory, noncommutative geometry, algebraic geometry, and mathematical physics.

Reviewer: Werner Kleinert (Berlin)

MSC:

14A22 Noncommutative algebraic geometry
14G10 Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture)
14C35 Applications of methods of algebraic $K$-theory in algebraic geometry
14C25 Algebraic cycles
14G25 Global ground fields in algebraic geometry
14F43 Other algebro-geometric (co)homologies (e.g., intersection, equivariant, Lawson, Deligne (co)homologies)
11G40 $L$-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture
81T15 Perturbative methods of renormalization applied to problems in quantum field theory

Keywords:
varieties over global fields; Riemann hypothesis; $K$-theory; quantum field theory; $L$-series

Full Text: DOI

References:

Exp. 19, 1969/1970


[53] Tate, J., Number theoretic background, (), 3-26, Part 2 · Zbl 0422.12007

[54] A. Weil, Sur les formules explicites de la théorie des nombres premiers, Oeuvres complètes 2, 48-62


This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.