zbMATH Open
THE FIRST RESOURCE FOR MATHEMATICS

Bertrand, D.; Zudilin, W.

On the transcendence degree of the differential field generated by Siegel modular forms. (English) Zbl 1130.11020

J. Reine Angew. Math. 554, 47-68 (2003).

Let q be a positive integer and k be an algebraically closed subfield of $\mathbb{C}$, $\mathfrak{H}_g$ Siegel half space of degree g, $\tau = (\tau_{jl})$ a generic point on $\mathfrak{H}_g$, and Γ a congruence subgroup of the symplectic group $\mathrm{Sp}_{2g}K$. The field of modular functions $K = K(\Gamma, k)$ has the form $K = k(\lambda)$, where $\lambda = \{\lambda_1, \dots, \lambda_N\}$ is a set of modular function relative to Γ, whose first $n = \mathrm{trdeg}(K/k)$ elements are algebraically independent. Set $\delta = \{\delta_{jl}, 1 \le j \le l \le q\}$ where $\delta_{ij} = \frac{1}{2\pi i}\frac{\partial}{\partial \tau_{jl}}, 1 \le j < l < g, \delta_{jj} = \frac{1}{\pi i}\frac{\partial}{\partial \tau_{jj}}, 1 \le j \le g$. Let $M = M(\Gamma, k) = k\langle \lambda_1, \dots, \lambda_N \rangle$ be the δ-differential field generated by K.

The main result of the paper (Theorem 1) asserts that M is a finite extension of the field generated over K by the δ-partial derivatives $\lambda_1, \dots, \lambda_N$ of order ≤ 2 and has the transcendence degree $2g^2 + g$, M and $\mathbb{C}(\tau)$ are linearly disjoint over k, and $\mathrm{trdeg}(M(2\pi i\tau)/h) = \frac{g(5g+3)}{2}$.

Taking for Γ the theta group $\Gamma_{4,8}$ of level 4,8, and $\lambda = \{\vartheta_a/\vartheta_0, a \in (\mathbb{Z}/2\mathbb{Z})^{2g}\}$ this result can be stated in the following more precise theorem:

Theorem 2. The δ-derivative of order ≤ 2 of the modular functions $\{\vartheta_a/\vartheta_0, a \in (\mathbb{Z}/2\mathbb{Z})^{2g}\}$ generate over k a δ-stable field $M(\Gamma_{4,8}, k)$ of transcendence degree $2g^2 + g$ over k, over which ϑ_0 is algebraic. F

inally, this last result is sharpened in the cases $g = 1, 2$. This is done in the final section of the paper. The proofs use the Picard-Fuchs and Picard-Vessiot theories, differential forms and the explicit formulae on derivatives of theta functions.

Reviewer: Mykola Ya. Komarnytskyy (Lviv)

MSC:

11F46 Siegel modular groups; Siegel and Hilbert-Siegel modular and automorphic forms
12H05 Differential algebra

Keywords:

Siegel modular forms; theta functions; field of modular functions; δ-derivatives; Picard-Fuchs extension; Picard-Vessiot extensions

Full Text: DOI arXiv

References:

[1] Compos. Math. 82 pp 1– (1992)
[2] Andre Y., Math. 476 pp 95– (1996)
B D. Bertrand, Endomorphismes de groupes algebriques; applications arithmetiques, Approximations diophantiennes et nombres transcendants (Colloque de Luminy, 1982), D. Bertrand, M. Waldschmidt, eds., Progr. Math. 31, Birkh user, Boston-Basel-Stuttgart (1983), 1-45.
[3] Deligne P., . Sci. Publ. Math. 40 pp 5– (1972)
D2 P. Deligne, Hodge cycles on abelian varieties, Notes by J. S. Milne, Hodge Cycles, Motives, and Shimura Varieties, Lect. Notes Math. 900, Springer, Berlin (1982), 9-100.
I J. Igusa, Theta Functions, Grundl. Math. Wiss.194, Springer, Berlin-Heidelberg-New York 1972. · Zbl 0251.14016
Ka K. N. Katz, p-adic properties of modular schemes and modular forms, Modular Functions of one Variable III, Proc. Internat. Summer School (Univ. Antwerp, 1972), Lect. Notes Math. 350, Springer, Berlin (1973), 69-190.
Kr A. Krazer, Lehrbuch der Thetafunktionen, Leipzig1903; Reprint, Chelsea Publ., New York 1970.
[4] Lang S., Encyclop. Math. Sci. 60 pp 1991–
[5] J. Austral. Math. Soc. 10 pp 445– (1969)

MF D. Mumford, J. Fogarty, Geometric Invariant Theory, 2nd edition, Ergeb. Math. Grenzgeb.34, Springer, Berlin-New York 1982. · Zbl 0504.14008

O Y. Ohyama, Di erential equations of theta constants of genus two, Algebraic Analysis of Singular Perturbations (Kyoto 1996), Su rikaisekikenkyu sho Ko kyu roku 968, Kyoto Univ., Kyoto (1996), 96-103.

[6] Parsin A. N., Trudy Mat. Inst. Steklov. 132 pp 211– (1973)

[7] R., J. Indian Math. Soc. 20 pp 103– (1956)

[8] H., Abh. Math. Semin. Univ. Hamburg 38 pp 168– (1972)

S G. Shimura, Abelian Varieties with Complex Multiplication and Modular Functions, Princeton Math. Ser.46, Princeton Univ. Press, Princeton, NJ, 1998. · Zbl 0908.11023

[9] Mat. Sbornik 192 pp 8– (2000)