Thomas, R. P.
Notes on GIT and symplectic reduction for bundles and varieties. (English) Zbl 1132.14043

This is a survey that presents the recent advances about the problem of constant scalar curvature metrics on a projective manifold from the point of view of complex algebraic geometry.

First of all, the author gives a review of what is D. Mumford’s theory for taking quotient of projective or affine varieties by reductive complex linear algebraic groups. This is the famous geometric invariant theory, G.I.T in short. This part is very pedagogical with a lot of examples, simple but fundamental. This gives a precise idea of what is G.I.T stability or symplectic reduction and their relationship through the Kempf-Ness theorem which is explained in details. This part may interest any graduate students.

The rest of the paper is dedicated to the so-called Yau-Tian-Donaldson conjecture over smooth projective manifolds. Two decades ago, S.T. Yau conjectured that the existence of a constant scalar curvature metric in a Kähler class (cscK metric in short) is equivalent to a certain algebraic notion of ‘stability’. This is a fundamental problem in Kähler geometry, generalizing in a certain sense the uniformisation theorem for Riemann surfaces. This can also be considered as the analogue of the Kobayashi-Hitchin correspondence for manifolds. Let’s recall that the Kobayashi-Hitchin correspondence proved by Donaldson, Uhlenbeck and Yau in the 80’s, states that an irreducible holomorphic vector bundle on a compact Kähler manifold is Mumford stable if and only if there exists, on this bundle, a smooth metric solution to the Hermitian-Einstein equation.

The notion of ‘stability’ was later precised by G. Tian and S.K. Donaldson who called it K-stability. Other mathematicians have worked on different notions of stability, for instance asymptotic Chow (or Hilbert) stability. The author presents these notions and the relationship between them from the point of view of G.I.T. He explains how Chow-stability is related to the existence of balanced metrics. On another hand, these algebraic metrics are proved to exist when there is a cscK metric by S. K. Donaldson’s theory [J. Differ. Geom. 59, 479–522 (2001; Zbl 1052.32017)]. This lead to purely infinite dimensional formal symplectic quotient formulation of the cscK problem. The quantization of the cscK equation and Donaldson’s double quotient is explained in details in a formal way (another good reference is [O. Biquard, Bourbaki seminar. Volume 2004/2005. SMF. Astérisque 307, Exp. No. 938 1–31 (2006; Zbl 1142.32010]) for the technical details). The description of what happens for bundles enlightens the picture that we have for the moment of the situation. At the time we write these lines, only the implication “existence of a cscK metric” ⇒ “K-semi-stability” and “asymptotic Chow stability” is known (actually a very recent preprint of J. Stoppa shows “existence of a cscK metric” ⇒ “K-stability”). Note that T. Mabuchi has related asymptotic Chow stability and Hilbert stability and that there have been progress on toric surfaces by Donaldson.

Finally, the author describes his work with J. Ross about slope stability for varieties. Roughly speaking they modeled from the world of bundles a notion of Mumford stability for a couple \((M, L)\) where \(M\) is a projective manifold and \(L\) an ample line bundle. This notion is weaker than K-stability but on the other hand is easier to check. It has lead recently to detect non cscK classes for Kähler-Einstein manifolds [J. Ross, Invent. Math. 165, No. 1, 153–162 (2006; Zbl 1107.14011)]. The author shows that slope stability comes from K-stability by specifying certain test-configurations. This gives at the same time a more precise idea of what means geometrically the subtle notion of K-stability.

Probably the only missing objects which have not been described in that paper are Nadel’s ideal sheaves and the singularities that appear when we apply test-configurations (both are probably related in the case of \(L = ± K_M\), i.e one is considering the Kähler-Einstein equation). But this survey is extremely well written, covers a lot of aspects of the cscK problem, and we definitively recommend it for any people interested by the subject or the notion of G.I.T. It also underlines that the Yau-Tian-Donaldson conjecture is living on the border of different worlds: symplectic geometry, algebraic geometry and complex geometry.
For the entire collection see [Zbl 1117.53003].

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MSC:

14L24 Geometric invariant theory
53D20 Momentum maps; symplectic reduction
32L10 Sheaves and cohomology of sections of holomorphic vector bundles, general results
53C25 Special Riemannian manifolds (Einstein, Sasakian, etc.)

Keywords:

GIT; symplectic reduction; Kobayashi-Hitchin correspondence; polarized varieties; test configurations; constant scalar curvature; moment map; stability; Chow; Hilbert; Einstein