Pucci, Patrizia; Serrin, James
The maximum principle. (English) Zbl 1134.35001
Progress in Nonlinear Differential Equations and Their Applications 73. Basel: Birkhäuser (ISBN 978-3-7643-8144-8/hbk). x, 235 p. (2007).

The Maximum Principle is a powerful method in Nonlinear Analysis, as an important tool in the qualitative analysis of many classes of nonlinear PDEs. In its simplest form, the Maximum Principle asserts that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a superharmonic function on a bounded domain $\Omega \subset \mathbb{R}^N$ such that $u \geq 0$ on $\partial\Omega$ then $u \geq 0$ in Ω. Moreover, the following alternative holds: either $u \equiv 0$ in Ω or $u > 0$ in Ω. The Maximum Principle goes back to the papers by E. Hopf and has been extended by Stampacchia for linear perturbations of the Laplace operator and by Vázquez in the case of nonlinear perturbations of the Laplace operator, in connection with a certain growth of the nonlinearity around the origin. The authors of this volume have important and deep contributions in the understanding of the Maximum Principle, as well as in its generalization to wide classes of differential operators.

This book is composed of eight chapters the first one introducing the main notations and some preliminary results. Chapter 2 deals with various results on tangency and comparison theorems for elliptic inequalities. The authors recall some of the main contributions of Hopf, tangency theorems via Harnack's inequality, as well as several comparison theorems for divergence structure inequalities. Chapter 3 is concerned with the generalization of these results for more general operators and the main results include the "thin set" maximum principle and maximum principles for weakly singular inequalities and strongly singular inequalities.

Chapter 4 is a digression from the earlier emphasis on tangency, comparison and maximum principles, dealing instead with two-point boundary value problems for nonlinear ordinary differential equations. This work is preliminary to the strong maximum principles of Chapter 5, but also has ramifications in some unexpected byways. In particular, there are intimate connections with the exterior Dirichlet boundary value problem and with the existence of dead cores at infinity. The next chapter is concerned with maximum principles for the complete quasilinear divergence inequality

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) \geq 0$$

under general assumptions on A and B.

In Chapter 7 the authors obtain important applications, such as: (i) local boundedness properties, (ii) Harnack inequalities, (iii) a generalization of De Giorgi's theorem on Hölder's continuity, in connection with the John-Nirenberg inequality. Further applications are deduced in the last chapter of this volume.

Throughout this book the framework is very general and the results can be applied to important classes of elliptic equations and inequality problems with deep physical or geometric interest, including: the prescribed mean curvature equation, the stereographic projection equation, subsonic gas dynamics, the general equation of radiative cooling, the Monge-Ampère equation, or Calabi's equation.

This book is certain to become a source of inspiration for every researcher in nonlinear analysis. While the level of difficulty is uneven, the book is full of insights and useful tidbits. The present text succeeds admirably, in the reviewer's opinion, in introducing its difficult subject at a level appropriate for preparing future workers in the field.

Reviewer: Vicenţiu D. Rădulescu (Craiova)

MSC:

- 35-02 Research exposition (monographs, survey articles) pertaining to partial differential equations
- 35B50 Maximum principles in context of PDEs
- 35A05 General existence and uniqueness theorems (PDE) (MSC2000)
- 35B05 Oscillation, zeros of solutions, mean value theorems, etc. in context of PDEs
- 35J15 Second-order elliptic equations
- 35J60 Nonlinear elliptic equations
- 35J70 Degenerate elliptic equations

Keywords:

maximum principle; elliptic operator; dead core; Harnack inequality