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Pillowcases and quasimodular forms. (English) Zbl 1136.14039
Ginzburg, Victor (ed.), Algebraic geometry and number theory. In Honor of Vladimir Drinfeld's 50th birthday. Basel: Birkhäuser (ISBN 978-0-8176-4471-0/hbk). Progress in Mathematics 253, 1-25 (2006).

This paper gives a characterization for the generating functions enumerating branched coverings of the pillowcase orbifold in terms of level 2 quasimodular forms. The pillowcase orbifold $\mathbb{B}$ is obtained as the quotient of a complex torus $\mathbb{T}^2$ by the involution ± 1. From the construction, $\mathbb{B}$ has four $(\mathbb{Z}/2\mathbb{Z})$ orbifold points. The map $\mathbb{T}^2 \to \mathbb{B}$ is essentially the Weierstrass $\wp$-function.

Let μ be a partition and ν is a partition of an even number into odd parts. The paper is concerned with the enumeration of covers $\pi : C \to \mathbb{B}$ of degree $2d$ with the following specified ramification data: Over $0 \in \mathbb{B}$ π has ramification profile $(\nu, 2^{d-|\nu|/2})$ and over other orbifold points ramification profile (2^d); in addition, over some $\ell(\mu)$ points of $\mathbb{B}$ ramification profile $(\mu_i, 1^{2d-\mu_i})$ with $\ell(\mu)$ denoting the number of parts in μ, and unramified elsewhere. The genus of C is determined by this data.

Define the generating function

$$Z(\mu,\nu;q) := \sum_{\pi} \frac{q^{deg(\pi)}}{|\mathrm{Aut}(\pi)|}$$

where the sum ranges over all inequivalent covers with ramification data (μ, ν). If $\mu = \nu = \emptyset$, then $Z(\emptyset, \emptyset; q) = \prod_n (1-q^{2n})^{-1/2}$. Define a normalized generating function

$$Z'(\mu,\nu;q) = \frac{Z(\mu,\nu;q)}{Z(\emptyset,\emptyset;q)}.$$

The main result of the paper is the following characterization theorem for $Z'(\mu,\nu;q)$.

Theorem. The series $Z'(\mu,\nu;q)$ is a polynomial in $E_2(q^2), E_2(q^4)$ and $E_4(q^4)$ of weight $|\mu| + \ell(\mu) + |\nu|/2$.

Here $E_{2k}(q)$ is the classical Eisenstein series defined by

$$E_{2k}(q) = \frac{\zeta(1-2k)}{2} + \sum_{n=1}^{\infty} (\sum_{d|n} d^{2k-1})\, q^n$$

and it is known that $E_{2k}(q^N)$ is a quasimodular form of weight $2k$ for the modular group $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$.

Quasimodular forms occur, for instance, as coefficients of the expansion of the odd genus one theta-function

$$\theta(x) = (x^{1/2} - x^{-1/2}) \prod_{i=1}^{\infty} \frac{(1-q^i x)(1-q^i/x)}{(1-q^i)^2}$$

at the origin $x = 1$. The theorem is proved identifying $Z'(\mu,\nu;q)$ with derivatives of $\theta(x)$ at $x = \pm 1$, which explain the appearance of quasimodular forms.

The proof of the main result is an application of the method on enumerating branched coverings in terms of irreducible characters of the symmetric group, developed, for instance, in [*A. Okounkov* and *R. Pandharipande*, Ann. Math. (2) 163, No. 2, 517–560 (2006; Zbl 1105.14076)]. This method yields the following formula for $Z(\mu,\nu;q)$:

$$Z(\mu,\nu;q) = \sum_{\lambda} q^{|\lambda|/2} (\frac{\dim\lambda}{|\lambda|!})^2\, \mathbf{f}_{\nu,2,2,\dots}(\lambda) \mathbf{f}_{2,2,\dots}(\lambda)^3 \prod_i \mathbf{f}_{\mu_i}(\lambda) \qquad (*)$$

where the sum runs over all partitions. Here λ is a representation of the symmetric group, $\dim\lambda$ is its dimension, and $\mathbf{f}_\eta(\lambda)$ is the central character of an element with cycle type η in the representation λ (for simplicity, $\mathbf{f}_k = \mathbf{f}_{k,1,1,\dots}$.) The functions $\mathbf{f}_k$ belong to the algebra generated by the functions

$$\mathbf{p}_k(\lambda) = (1-2^{-k})\zeta(-k) + \sum_i [(\lambda_i - i + \frac{1}{2)})^k - (-i + \frac{1}{2})^k]$$

by the result of *S. Kerov* and *G. Olshanski* [C. R. Acad. Sci., Paris, Sér. I 319, No. 2, 121–126 (1994; Zbl 0830.20028)]. The authors generalize this result by enlarging the algebra of functions and work with the generating function

$$\mathbf{e}(\lambda, z) := \sum_i e^{z(\lambda_i - i + 1/2)} = \frac{1}{z} + \sum_k \mathbf{p}_k(\lambda) \frac{z^k}{k!},$$

and define

$$\bar{\mathbf{p}}_k(\lambda) = ik![z^k]\mathbf{e}(\lambda, z + \pi\, i)$$

$$= \sum_i [(-1)^{\lambda_i - i+1}(\lambda_i - i + \frac{1}{2})^k - (-1)^{-i+1}(-i + \frac{1}{2})^k] + \text{constant}.$$

Consider the $\mathbb{Q}$-algebra, $\bar{\Lambda} = \mathbb{Q}[\mathbf{p}_k,\, \bar{\mathbf{p}}_k]_{k\geq 1}$ generated by $\mathbf{p}_k$ and $\bar{\mathbf{p}}_k$ $(k \geq 1)$, and equipped with a weight filtration given by $wt\, \mathbf{p}_k = k+1$, $wt\, \bar{\mathbf{p}}_k = k$. The formula (*) is rearranged by introducing some notations: For a balanced partition λ (i.e., $\bar{\mathbf{p}}_0(\lambda) = 1/2$), define

$$\mathbf{g}_\nu(\lambda) = \mathbf{f}_{\nu,2,2,\dots}(\lambda)/\mathbf{f}_{2,2,\dots}(\lambda)$$

and introduce the pillowcase weight

$$\mathbf{w}(\lambda) = (\frac{\dim \lambda}{|\lambda|!})^2 \mathbf{f}_{2,2,\dots}(\lambda)^4.$$

Then the main result is reformulated as follows:

Theorem A. There is a unique function in $\bar{\Lambda}$ of weight $|\nu|/2$ which coincides with $\mathbf{g}_\nu(\lambda)$ when restricted to balanced partition.

Theorem B. For a function $F \in \bar{\Lambda}$, let

$$< F >_{\mathbf{w}} := \frac{1}{Z(\emptyset, \emptyset; q)} \sum_\lambda q^{|\lambda|} \mathbf{w}(\lambda) F(\lambda)$$

be the average of F. Then $< F >_{\mathbf{w}}$ is a polynomial in $E_2(q^2), E_2(q^4)$ and $E_4(q^4)$ of weight $wt\, F$.

The proof of Theorem B is done determining an operator formula for the n-point function

$$F(x_1, \dots, x_n) := < \prod \mathbf{e}(\lambda, \ln x_i) >_{\mathbf{w}},$$

and then finding a formula for it in terms of traces of the energy operator. Finally $F(x_1, \dots, x_n)$ has an expression in terms of products of the odd genus one theta function, from which quasimodular forms E_2 arise.

An application of the main result is to volumes of certain moduli spaces. Let $Z^\circ(\mu, \nu; q)$ denote the generating function counting connected covers. The asymptotic behavior of $Z^\circ(\mu, \nu; q)$ as $q \to 1$ contains information on the volume of certain moduli space of quadratic differentials. The moduli space in question here is the moduli space $Q(\mu, \nu)$ of pairs (Σ, ϕ) where ϕ is a quadratic differential on a curve Σ with zero of multiplicities $[\nu_i - 2, 2\mu_i - 2]$. Consider the subset $Q_1(\mu, \nu)$ of $\mathbb{Q}(\mu, \nu)$ satisfying a certain area condition. Then

$$\rho(\mathbb{Q}_1(\mu, \nu)) = \lim_{D\to\infty} D^{-\dim_{\mathbb{C}} \mathbb{Q}(\mu,\nu)} \sum_{d=1}^{2D} \mathrm{Cov}_d^0(\mu, \nu)$$

where $\mathrm{Cov}^0(\mu, \nu)$ is the number of inequivalent degree d connected covers of $C \to \mathbb{B}$, and ρ is a measure on $\mathbb{Q}_1(\mu, \nu)$. Then the volume of $Q_1(\mu, \nu)$ can be read off from the $q \to 1$ asymptotics of the connected generating function $Z^\circ(\mu, \nu; q)$.

For the entire collection see [Zbl 1113.00007].

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MSC:

14N10	Enumerative problems (combinatorial problems) in algebraic geometry
14N30	Adjunction problems
11F23	Relations with algebraic geometry and topology
14N35	Gromov-Witten invariants, quantum cohomology, Gopakumar-Vafa invariants, Donaldson-Thomas invariants (algebro-geometric aspects)

Keywords:

pillowcase cover; quadimodular forms; generating function; character sum

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