
From the text: A major consideration we had in writing this survey was to make it accessible to mathematicians as well as to computer scientists, since expander graphs, the protagonists of our story, come up in numerous and often surprising contexts in both fields. ...

Expander graphs were first defined by L. A. Bassalygo and M. S. Pinsker [Probl. Inf. Transm. 9, 64–66 (1974); translation from Probl. Peredachi Inf. 9, No. 1, 84–87 (1973; Zbl 0327.94051)], and their existence first proved by Pinsker in the early 1970s [On the complexity of a concentrator. In: Proc. 7th Int. Teletraffic Conference, Stockholm 1973, 318/1–318/4 (1973)]. The property of being an expander seems significant in many of these mathematical, computational and physical contexts. It is not surprising that expanders are useful in the design and analysis of communication networks. What is less obvious is that expanders have surprising utility in other computational settings such as in the theory of error correcting codes and the theory of pseudorandomness.

In mathematics, we will encounter e.g. their role in the study of metric embeddings, and in particular in work around the Baum-Connes Conjecture. Expansion is closely related to the convergence rates of Markov chains, and so they play a key role in the study of Monte-Carlo algorithms in statistical mechanics and in a host of practical computational applications. The list of such interesting and fruitful connections goes on and on with so many applications we will not even be able to mention. This universality of expanders is becoming more evident as more connections are discovered. It transpires that expansion is a fundamental mathematical concept, well deserving to be thoroughly investigated on its own.

In hindsight, one reason that expanders are so ubiquitous is that their very definition can be given in at least three languages: combinatorial/geometric, probabilistic and algebraic. Combinatorially, expanders are graphs which are highly connected; to disconnect a large part of the graph, one has to sever many edges. Equivalently, using the geometric notion of isoperimetry, every set of vertices has a (relatively) very large boundary. From the probabilistic viewpoint, one considers the natural random walk on a graph, in which we have a token on a vertex, that moves at every step to a random neighboring vertex, chosen uniformly and independently. Expanders are graphs for which this process converges to its limiting distribution as rapidly as possible. Algebraically, one can consider the Laplace operator on the graph and its spectrum. From this perspective, expanders are graphs in which the first positive eigenvalue (of their Laplace operator) is bounded away from zero.

The study of expanders leads in different directions. There are structural problems: what are the best bounds on the various expansion parameters, and how do they relate to each other and to other graph invariants? There are problems concerning explicit constructions: how to efficiently generate expanders with given parameters. These are extremely important for applications. There are algorithmic problems – given a graph, test if it is an expander with given parameters. Finally, there is the problem of understanding the relation of expansion with other mathematical notions, and the application of expanders to practical and theoretical problems.

In the past four decades, a great amount of research has been done on these topics, resulting in a wide-ranging body of knowledge. In this survey, we could not hope to cover even a fraction of it. We have tried to make the presentation as broad as possible, touching on the various research directions mentioned above. Even what we do cover is of course incomplete, and we try to give the relevant references for more comprehensive coverage. We have also tried to mention in each section related research which we are not covering at all and to reference some of this as well.

The selection of material naturally reflects our interests and biases. It is rather diverse and can be read in different orders, according to the reader’s taste and interests.


This article evolved from lecture notes for a course on expanders taught at the Hebrew University, Israel, in 2003 by Nati Linial and Avi Wigderson.

Contents:

1. The magical mystery tour: Three motivating problems: Hardness results for linear transformations, Construction of good error correcting codes, Deterministic error amplification for RP; Magical graphs; The three solutions: A super concentrator with \(O(n)\) edges, Construction of good error correcting codes, Deterministic error amplification for RP.

2. Graph expansion and eigenvalues: Edge expansion and a combinatorial definition of expanders; Examples of expander graphs; Graph spectrum and an algebraic definition of expansion; The Expander Mixing Lemma: How big can the spectral gap be? Four perspectives on expansion and how they compare: Extremal problems, Typical behavior, Explicit constructions, Algorithms, Comparisons.

3. Random walks on expander graphs: Rapid mixing of walk: Convergence in the \(l_1\) and \(l_2\) norms, Convergence in entropy; Random walks resemble independent sampling; Applications: Efficient error reduction in probabilistic algorithms, Hardness of approximating maximum clique size.

4. A geometric view of expander graphs: The classical isoperimetric problem; Graph isoperimetric problems, Example: The discrete cube; The Margulis construction: The discrete Laplacian; The Cheeger constant and inequality; Expansion and the spectral gap: Large spectral gap implies high expansion, High expansion implies large spectral gap; Expansion of small sets: Connection with the spectral gap, Typical behavior; Expansion in hypergraphs?.

5. Extremal problems on spectrum and expansion: The d-regular tree: The expansion of \(T_d\), The spectrum of \(T_d\); The Alon-Boppana lower bound: Statement of the theorem, Proof I: Counting closed walks in \(T_d\), Proof II: Using spherical functions; Extensions of the Alon-Boppana theorem; Ramanujan graphs.


7. The spectrum of random graphs: The bulk of the spectrum; The extreme eigenvalues: An illustration of the trace method; Variations on a theme: Back to the irregular case, Are most regular graphs Ramanujan?, More on random lifts, The eigenvectors.

8. The Margulis construction: A detour into harmonic analysis: Characters; Back to the proof.

9. The zig-zag product: Introduction; Construction of an expander family using zig-zag; Definition and analysis of the zig-zag product; Entropy analysis; An application to complexity theory: SL = L.

10. Lossless conductors and expanders: Conductors and lossless expanders; Conductors; Lossless expanders; The construction; The zig-zag product for conductors; Proof of the main theorem; Final comments.

11. Cayley expander graphs: Representations of finite groups; Representations and irreducible representations; Schreier graphs; Kazhdan constant and expansion of Cayley graphs; The replacement product and semidirect product; Constructing expander families by iterated semidirect products; Cayley expanders from group rings; Cayley expanders from iterated wreath products; Expansion is not a group property; Hypercontractive inequalities in groups.

12. Error correcting codes: Definition of error correcting codes; Linear code; Asymptotic bounds; Lower bounds on size: The Gilbert-Varshamov bound; Upper bounds: Sphere packing and linear programming; Codes from graphs; Efficient asymptotically good codes from lossless expanders.

13. Metric embedding: Basic definitions; Finding the minimal \(l_2\) distortion; Distortion bounds via semidefinite duality; Embedding the cube into \(l_2\); Embedding expander graphs into \(l_2\); Algorithms for cut problems via embeddings; A glimpse into the bigger picture.

Reviewer: Olaf Ninnemann (Uffing am Staffelsee)
References:


[73] Andrew Granville, It is easy to determine whether a given integer is prime, Bull. Amer. Math. Soc. (N.S.) 42 (2005), no. 1, 33 – 38. - Zbl 1110.11002


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S. Goory, A. Magen, and T. Pitassi. Monotone circuits for the majority function. 10th International Workshop on Randomization and Computation (RANDOM), 2006.


Satyanarayana V. Lokam, Spectral methods for matrix rigidity with applications to size-depth trade-offs and communication...


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