Lev, Vsevolod F.

Distribution of points on arcs. (English) Zbl 1156.11310

Integers 5, No. 2, Paper A11, 6 p. (2005).

Let $z_1, \dots, z_N$ be complex numbers situated on the unit circle $\mathbb{U} := \{z \in \mathbb{C} \mid |z| = 1\}$, and write $S := z_1 + \dots + z_N$. In 1962 *G. A. Freiman* [Izv. Vyssh. Uchebn. Zaved., Mat. 1962, No. 6(31), 131–144 (1962; Zbl 0171.00803)] established a lemma showing that if $z_1, \dots, z_N \in \mathbb{U}$ are "uniformly distributed on arcs of length π", then the sum $z_1 + \dots + z_N$ is small in absolute value, $S \leq 2n - N$. The assumption "any open arc of $\mathbb{U}$ of length π contains at most n of the numbers $z_1, \dots, z_N$" implies readily that $N \leq 2n$.

In practice, an analog of Lemma 1 is needed with the arcs of length π replaced by arcs of other prescribed lengths; see for instance [*B. Green* and *I. Z. Ruzsa*, Bull. Lond. Math. Soc. 38, No. 1, 43–52 (2006; Zbl 1155.11307), *T. Schoen*, Integers 3, Paper A17, 6 p., electronic only (2003; Zbl 1089.11009)]. The author generalizes Freiman's result as follows:

Theorem 1: Suppose that any open arc of length $\phi \in (0, \pi]$ of the unit circle contains at most n of the numbers $z_1, \dots, z_N$. Then

$$|S| \leq 2n - N + 2(N - n)\cos(\phi/2).$$

Theorem 1 is sharp, at least in the range $N/2 \leq n \leq N$: the bound is attained, for instance, if $2n - N$ of the numbers z_j equal one, $N - n$ equal $\exp(i\phi/2)$, and $N - n$ equal $\exp(-i\phi/2)$.

Theorem 2: Suppose that any open arc of length π of the unit circle contains at most n of the numbers $z_1, \dots, z_N$ and suppose, in addition, that for any $1 \leq i < j \leq N$ the length of the (shortest) arc between z_i and z_j is at least $\delta > 0$. Then

$$|S| \leq \frac{\sin(n - N/2)\delta}{\sin \delta/2}$$

provided that $n\delta \leq \pi$. Theorem 2 is also sharp.

Finally the author notices that from Theorem 1 one can deduce the following more general result.

Theorem 1'. Let $\lambda \leq 1/2$ and ν be positive real numbers and let μ be a probabilistic measure on the torus group $\mathbb{R}/\mathbb{Z}$. Suppose that $\mu(I) \leq \nu$ for any open interval $I \subseteq \mathbb{R}/\mathbb{Z}$ of length $|I| = \lambda$. Then

$$\left| \int_{\mathbb{R}/\mathbb{Z}} \exp(it)\, d\mu \right| \leq 2\nu - 1 + 2(1 - \nu)\cos(\pi\lambda).$$

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MSC:

11J71 Distribution modulo one
11K36 Well-distributed sequences and other variations

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