Katz, Gabriel

Curves in cages: an algebro-geometric zoo. (English) Zbl 1156.14019

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The author defines a $(d \times e)$-cage as the intersection of two unions $\mathcal{D}$ and $\mathcal{E}$ of d (resp. e) lines in the real or complex projective plane. The lines in $\mathcal{D}$ are called "red", the lines in $\mathcal{E}$ are called "blue". The author studies algebraic curves of degree k through "many" nodes of the cage.

A subset $\mathcal{A}$ of the cage nodes is called *quasi-triangular* if every blue line contains between $d-e+1$ and d elements of $\mathcal{A}$ and each value is attained exactly once; it is called *supra-quasi-triangular*, if the node number is between $d-e+2$ and d and each value between $d-e+2$ and $d-1$ is attained exactly once (then the number d is attained twice). The following main results hold true:

1. "If a curve in $\mathbb{P}^2$ of degree d passes through a supra-quasi-triangular set $\mathcal{A}$ of nodes of a $(d \times e)$-cage with $d \geq e$, then it passes through all nodes of the cage."

2. "No curve of degree less then e can pass through a quasi-triangular set of nodes of a $(d \times e)$-cage when $d \geq e$."

3. Consider a $(d \times d)$-cage K, let R and B be the products of linear polynomials that describe its red and blue lines, respectively, and denote by $S_{[\lambda:\mu]}$ the curve defined by $\lambda R + \mu B$. Then every direction t at one of the cage nodes p defines a unique curve $S_{[\lambda:\mu]}$ that is tangent to t in p. Moreover, any curve of degree d through a supra-quasi-triangular subset $\mathcal{A}$ of K having t as tangent direction at $p \in \mathcal{A}$ contains all nodes of K.

4. If a polygon $\mathcal{D}$ with $2d$ sides, alternately colored in red and blue, is inscribed into a quadratic curve $\mathcal{Q}$ it generates a cage K such that the d^2-d nodes of $K \setminus \mathcal{Q}$ lie on a curve $\mathcal{Q}_\star$ of degree $d-2$ or less. Generically, $\mathcal{Q}_\star$ is unique.

Throughout the text the author highlights possible generalisations and relations to classical results such as the Theorem of Pascal on hexagons inscribed into a quadratic curve and Theorem of Chasles on complete intersections of cubic curves. Moreover, he pays attention to an elementary and entertaining treatment. Only in the concluding Section 4 he relates his findings to modern algebraic geometry by demonstrating how they can be derived from the Bacharach duality theorem.

Reviewer: Hans-Peter Schröcker (Innsbruck)

MSC:

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algebraic curve; complete intersection; Bacharach duality theorem

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