Madsen, Ib; Weiss, Michael

The stable moduli space of Riemann surfaces: Mumford's conjecture. (English)

Zbl 1156.14021
Ann. Math. (2) 165, No. 3, 843-941 (2007).

The moduli space $\mathcal{M}_g$ of Riemann surfaces of positive genus g is a quasi-projective variety of complex dimension $3g-3$. This is the space whose points correspond bijectively to isomorphism classes of nonsingular complex projective curves of genus g. Understanding the geometry or cohomology of these spaces for various g continues to be an attractive and challenging problem. Most challenging has been a conjecture of David Mumford dating from 1983 [*D. Mumford*, Arithmetic and geometry, Pap. dedic. I. R. Shafarevich, Vol. II: Geometry, Prog. Math. 36, 271-328 (1983; Zbl 0554.14008)] to the effect that the rational cohomology of the *stable* moduli space of Riemann surfaces is a polynomial algebra generated by certain classes κ_i of dimension $2i$ (the Mumford-Miller-Morita classes).

To state and explain the conjecture, one first considers the mapping class group Γ_g which consists of all isotopy classes of orientation preserving diffeomorphisms of a fixed compact oriented smooth surface F_g of positive genus g. This is a discrete group that acts properly discontinuously on the space of conformal structures on F_g up to isotopy (i.e. Teichmuller space $\mathcal{T}_g$). The quotient $\mathcal{T}_g/\Gamma_g$ is identified with $\mathcal{M}_g$ the moduli space. Because $\mathcal{T}_g$ is contractible and the action of Γ_g has finite stabilizers, it follows that $\mathcal{M}_g$ is rationally homotopy equivalent to the classifying space $B\Gamma_g$. From the topologist point of view, and at least with rational coefficients, it is enough to study $B\Gamma_g$.

By introducing boundary components and the associated mapping class groups $\Gamma_{g,b}$ where b is the number of boundary circles, it becomes possible to define maps $\Gamma_{g,b} \to \Gamma_{g+1,b}$ and $\Gamma_{g,b} \to \Gamma_{g,b-1}$ which at the level of classifying spaces induce isomorphisms in integral cohomology in degrees less than $g/2-1$ by fundamental work of Harer and Ivanov. The mapping telescope of the maps $B\Gamma_{g+i,b} \to B\Gamma_{g+i+1,b}$ yield in the limit a space whose cohomology is independent of b and which is written $B\Gamma_\infty$. For a given integer k, $H^k(B\Gamma_\infty;\mathbb{Q}) \cong H^k(B\Gamma_g;\mathbb{Q}) \cong H^k(\mathcal{M}_g;\mathbb{Q})$ for sufficiently large g. The Mumford conjecture is the statement that there is an algebra isomorphism

$$H^*(B\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\ldots]$$

Miller and Morita proved that the polynomial algebra on the right injects into the term on the left. In the paper at hand, the authors complete the proof of this conjecture by establishing the surjection. In fact they prove a much strengthened version of this conjecture due to *I. Madsen* and *U. Tillmann* [Invent. Math. 145, No. 3, 509–544 (2001; Zbl 1050.55007)].

Write $\Gamma_{g,2}$ as $\Gamma_{g,1+1}$ denoting the fact that 2-boundary components are now made into an incoming and an outgoing component. By gluing outgoing to incoming we make $\coprod_{g\geq 0} B\Gamma_{g,1+1}$ into a monoid and its "group completion" is given by $\mathbb{Z}\times B\Gamma_\infty^+$, where $B\Gamma_\infty^+$ is a space having the same homology groups as $B\Gamma_\infty$ but is simply-connected (that this space exists is due to the fact that Γ_g is a perfect group for $g\geq 3$). One major theorem of *U. Tillmann* states that this group completion is an infinite loop space [Invent. Math. 130, No.2, 257-275 (1997; Zbl 0891.55019)]. Madsen and Tillmann later conjectured that this infinite loop space is $\Omega^\infty\mathbb{C}P^\infty_{-1}$ the bottom space of a Thom spectrum associated to the anti-tautological bundles over complex projective spaces. Their conjecture involved the construction of an infinite loop map $\alpha_\infty : \mathbb{Z}\times B\Gamma^+_{\infty,2} \to \Omega^\infty\mathbb{C}P^\infty_{-1}$ and the assertion that this map is a homotopy equivalence. The main theorem of the paper under review is to establish this conjecture in its full strength. Since the rational cohomology of connected components of $\Omega^\infty\mathbb{C}P^\infty_{-1}$ is isomorphic to $\mathbb{Q}[\kappa_1,\kappa_2,\ldots]$, the Mumford conjecture becomes a consequence.

The proof provided here for the Madsen-Tillmann conjecture is a tour de force blending techniques from homotopy theory, algebraic geometry and sheaf theory. It is no easy task to even summarize it.

The first main idea is to describe the Madsen-Tillmann map α_∞ as a map between classifying spaces. Introduce sheaves $h\mathcal{V}$ and $\mathcal{V}$ defined on the category of smooth manifolds (the dimension $d=2$ corresponds to the Mumford conjecture) as follows : $h\mathcal{V}(X)$ for a manifold X is the set of pairs $(\pi,\hat{f})$ where π is a smooth submersion $E \to X$ with $(d+1)$-dimensional oriented fibers and $\hat{f}$ is an appropriate section of the vertical or fiberwise jet bundle over E (section 1). Similarly one defines $\mathcal{V}(X)$ (resp. $\mathcal{V}_c(X)$) as the set of pairs (π, f) with π a submersion $E \to X$ and f a smooth real function on E which is regular on fibers and such that the projection $(\pi, f) : E \to X \times \mathbb{R}$ is proper (resp. proper and having connected fibers). This is another way of describing a bundle of smooth closed d-manifolds on $X \times \mathbb{R}$ (see section 2). After introducing the notion of concordance of sheaves, one defines their representing spaces (or classifying spaces) denoted in our case by $|\mathcal{V}|$, $|\mathcal{V}_c|$ and $|h\mathcal{V}|$. Using the Ehresmann's fibration lemma and Phillip's submersion theorem, one is able to identify $|\mathcal{V}_c|$ and $|h\mathcal{V}|$ with $\mathbb{Z} \times B\Gamma^+_\infty$ and $\Omega^\infty \mathbb{C}P^\infty_{-1}$ respectively (in dimension $d = 2$). As one expects there is a natural map $|\mathcal{V}_c| \to |h\mathcal{V}|$ that models the Madsen-Tillmann map α_∞. To show homotopy equivalence, and hence establish the conjecture, the authors proceed by showing that both spaces are homotopy fibers of compatible and equivalent fibrations. This is explained next.

Define new sheaves $\mathcal{W}$ and $h\mathcal{W}$ by enlarging for each smooth closed manifold X the set $\mathcal{V}(X)$ to the set $\mathcal{W}(X)$ which consists of pairs (π, f) with π as before but with $f : E \to \mathbb{R}$ a fiberwise Morse function rather than a fiberwise regular function. There is a similar enlargement for $h\mathcal{V}(X)$ to $h\mathcal{W}(X)$. There is a so-called jet prolongation map $j : |\mathcal{W}| \longrightarrow |h\mathcal{W}|$ and one main theorem establishes that j is a homotopy equivalence (section 4). The main ingredient here is *V. A. Vassiliev*'s "first main theorem" on complements of discriminants [Complements of discriminants of smooth maps; Topology and applications. Translations of Math. Monographs 98, A.M.S., (Providence), RI. (1992; Zbl 0762.55001)].

The next key step is to show that the spaces $|h\mathcal{V}|$ and $|h\mathcal{W}|$ fit in a homotopy fibration sequence of infinite loop spaces $|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{loc}|$, where the latter classifying space is derived from an equally explicit sheaf construction. This homotopy fiber sequence is constructed in section 3 by identifying each of $h\mathcal{V}(X)$, $h\mathcal{W}(X)$ and $h\mathcal{W}_{loc}(X)$ on closed manifolds with a bordism construction and then identifying their classifying spaces, via the Thom-Pontryagin construction, with the corresponding infinite loop spaces of their Thom spectra. These Thom spectra are derived from the tautological bundle over the Grassmaniann which classify $(d+1)$-dimensional oriented vector bundles whose fibers are equipped with a Morse type map and with a linear embedding in $\mathbb{R}^{n+d+1}$ (section 3.1).

Analogously there is a space $|\mathcal{W}_{loc}|$ and maps $|\mathcal{W}| \to |\mathcal{W}_{loc}| \to |h\mathcal{W}_{loc}|$ such that the second map of this composite is a homotopy equivalence (section 3) and the homotopy fiber of the first map (in the two dimensional case) is the space $\mathbb{Z} \times B\Gamma^+_\infty$. The proof of this last claim takes up the largest bulk of the paper and is technically the most demanding part. It rests on three main steps: first giving homotopy colimit decompositions for the spaces $|\mathcal{W}|$ and $|\mathcal{W}_{loc}|$ (this is part of the very long section 5). The stratification pieces of these decompositions classify certain smooth surface bundles. It becomes necessary however to consider classifying spaces for surface bundles with connected fibers. Passing from not necessarily connected surfaces to connected surfaces is done through a well-organized surgery procedure in a way that doesn't change the homotopy type of the classifying space (section 6). Lastly Harer theorem on the homology stability of mapping class groups is used to finish off the identification of the homotopy fiber of $|\mathcal{W}| \to |\mathcal{W}_{loc}|$ (section 7).

Putting everything together, the homotopy equivalence α_∞ becomes a consequence of the fact that $|\mathcal{W}| \to |\mathcal{W}_{loc}|$ maps to $|h\mathcal{W}| \to |h\mathcal{W}_{loc}|$ through commuting homotopy equivalences and so the homotopy fiber of the first map; $\mathbb{Z} \times B\Gamma^+_\infty$, is homotopy equivalent to the fiber of the second map; $|h\mathcal{V}| \simeq \Omega^\infty \mathbb{C}P^\infty_{-1}$.

This long paper introduces a host of powerful ideas and techniques which will likely play in the future larger roles in all of algebraic topology and geometry. In section 4 and in Appendix A several pages are devoted to introducing and discussing the notion of sheaves taking values in the category of small categories, and a "classifying" space construction for such sheaves is given. In Appendix B, the homotopy invariance property of homotopy colimits is translated into the language of sheaves.

Reviewer: Sadok Kallel (Villeneuve d'Asq)

MSC:

14H10 Families, moduli of curves (algebraic)
55P35 Loop spaces

Keywords:

moduli space of curves; mapping class group; classifying spaces; thom spectra.

Full Text: DOI arXiv