Fock, V. V.; **Goncharov, A. B.**

Cluster χ-varieties, amalgamation, and Poisson-Lie groups. (English) Zbl 1162.22014

Ginzburg, Victor (ed.), Algebraic geometry and number theory. In Honor of Vladimir Drinfeld's 50th birthday. Basel: Birkhäuser (ISBN 978-0-8176-4471-0/hbk). Progress in Mathematics 253, 27-68 (2006).

Let G be a real semisimple Lie group with trivial center. For a given G, the authors define a family of Poisson varieties with certain additional structures, called cluster $\mathcal{X}$-varieties. These varieties are determined by combinatorial data similar to those (but different in some details) that are used for the definition of cluster algebras due to *S. Fomin* and *A. Zelevinsky* [J. Am. Math. Soc. 15 (2002; Zbl 1021.16017)]. These data consist of a collection of cluster seeds and mutations, that is, certain procedures producing from a given seed a new, mutated seed.

Now, for each cluster seed, we take an algebraic torus with a Poisson structure defined by this seed. Then we take some Poisson birational transformations between these tori, corresponding to all possible compositions of mutations, and glue these tori according to these transformations. The scheme obtained by this procedure is called a cluster X-variety. This construction was already defined by the authors in their earlier preprint (2003, arXiv:math.AG/0311245).

In the paper under review, they consider the braid semigroup $\mathfrak{B}$ associated with the Weyl group of a Lie group G, and define a cluster $\mathcal{X}$-variety $\mathcal{X}_B$ for each element $B \in \mathcal{B}$. Moreover, they define evaluation and multiplication maps between these varieties: $ev : \mathcal{X}_B \to G$ and $m : \mathcal{X}_{B_1} \times \mathcal{X}_{B_1} \to \mathcal{X}_{B_1 B_2}$. It is shown that these operations have a number of good properties (multiplication is associative, it commutes with the evaluation, they are Poisson maps), and express the multiplication map combinatorially, in terms of certain operations (amalgamation and defrosting) over seeds.

Then they define in a similar way cluster $\mathcal{X}$-varieties $\mathcal{X}_H$ associated with the elements $H \in \mathfrak{H}$ of the Hecke semigroup, obtained as a quotient of the braid group $\mathfrak{B}$, define evaluation and multiplication maps in this situation and show that they also enjoy the same properties. Moreover, both of these situations can be quantized: one can define quantum cluster $\mathcal{X}$-varieties for the braid and Hecke semigroups, multiplication maps, and show that they satisfy a quantum version of associativity.

In the Appendix to this paper, the authors consider the cluster $\mathcal{X}$-varieties related to the moduli spaces of configurations of triples of flags in the Lie groups of type B_2 and G_2, and provide an explicit description of the cluster structures on these varieties.

For the entire collection see [Zbl 1113.00007].

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MSC:

22E46	Semisimple Lie groups and their representations
05E15	Combinatorial aspects of groups and algebras (MSC2010)
53D17	Poisson manifolds; Poisson groupoids and algebroids
20G42	Quantum groups (quantized function algebras) and their representations

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Poisson–Lie groups; Poisson manifolds; clusters; cluster varieties; flag varieties