Borwein, Peter; Dobrowolski, Edward; Mossinghoff, Michael J.

Lehmer's problem for polynomials with odd coefficients. (English) Zbl 1172.11034

Ann. Math. (2) 166, No. 2, 347-366 (2007).

Let $f \in \mathbb{Z}[x]$ have odd coefficients, degree $n-1$ and no cyclotomic factors. Then the authors show that the Mahler measure $M(f)$ of f satisfies

$$\log M(f) \geq \frac{\log 5}{4}\left(1-\frac{1}{n}\right). \qquad (*)$$

This solves the well-known problem of D.H. Lehmer in the case of irreducible polynomials with odd coefficients. The result is generalised to polynomials f (degree $n-1$, no cyclotomic factors) whose coefficients are $\equiv 1 \pmod n$: in this case $\log M(f) \geq \log(m/2)(1-1/n)$. These results come from a more general theorem (Theorem 3.3) involving an auxiliary function F. In the case of such f having odd coefficients ($m=2$) this states that if F has integer coefficients and $\gcd(f(x), F(x^n)) = 1$ then

$$\log M(f) \geq \frac{\nu(F)\log 2 - \log \|F\|_\infty}{\deg F}\left(1-\frac{1}{n}\right).$$

Here $\|F\|_\infty = \max_{|z|=1} |F(z)|$ and $\nu(F)$ is the sum of the multiplicities of all factors of F that are 2^kth cyclotomic polynomials for some k. The corollary (*) is obtained by choosing $F(x) = (1+x^2)(1-x^2)^4$, the condition that f has no cyclotomic factors being required to ensure that $\gcd(f(x), F(x^n)) = 1$. The proof makes use of upper and lower bounds for the (nonzero) resultant $\mathrm{Res}(f(x), F(x^n))$.

Also, the conjecture of Schinzel and Zassenhaus is resolved for the case of polynomials f with odd coefficients (degree $n-1$, at least one noncyclotomic factor). The authors show that such an f has a root of modulus greater than $1+\log 3/n$. They also give the lower bound $1+\log(m-1)/n$ for the largest root of f when $m>2$ and the coefficients of f are all $\equiv 1 \pmod n$.

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MSC:

11R06 PV-numbers and generalizations; other special algebraic numbers; Mahler measure

11C08 Polynomials in number theory

11R09 Polynomials (irreducibility, etc.)

Keywords:

Lehmer's problem; Schinzel and Zassenhaus conjecture

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