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Homogeneous para-Kähler Einstein manifolds. (English. Russian original) Zbl 1179.53050


First the authors give a wide survey of known results about para-complex geometry. The second part of this paper is devoted to para-Kähler geometry.

A para-Kähler structure on a manifold $M$ is a pair $(g, K)$, where $g$ is pseudo-Riemannian metric and $K$ is a parallel skew-symmetric para-complex structure. A $2n$-dimensional pseudo-Riemannian manifold $(M, g)$ admits a para-Kähler structure $(g, K)$ if and only if its holonomy group is a subgroup of $\text{GL}_n(\mathbb{R}) \subset \text{SO}_{n,n} \subset \text{GL}_{2n}(\mathbb{R})$. If $(g, K)$ is a para-Kähler structure on $M$, then $\omega = g \circ K$ is a symplectic structure and the $\pm 1$-eigendistributions $T^\pm M$ of $K$ are two integrable $\omega$-Lagrangian distributions.

The authors derive some formulas for the curvature and Ricci curvature of a para-Kähler structure $(g, K)$ in terms of para-holomorphic coordinates and show that the Ricci tensor $S$ depends only on the determinant of the metric tensor $g_{\alpha \bar{\beta}}$. Next they consider a homogeneous manifold $(M = G/H, K, \text{vol})$ with an invariant para-complex structure $K$ and an invariant volume form $\text{vol}$. The authors prove that the pull-back $\pi^* \rho$ of the Ricci form $\rho = S \circ K$ of any para-Kähler structure $(g, K)$ is an exact form on $G$, that is $\pi^* \rho = d\psi$, where $\psi$ is a left-invariant 1-form called the Koszul form. Finally, they describe all invariant para-complex structures $K$ on $M = G/H$ in terms of fundamental gradations of the Lie algebra $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{h} := \text{Lie}(H)$ and show that they are consistent with any invariant symplectic structure $\omega$ on $G/H$. Moreover, the authors give a complete description of invariant para-Kähler-Einstein metrics on homogeneous manifolds of a semisimple Lie group.

Reviewer: Marian Hotloś (Wroclaw)

MSC:

53C25 Special Riemannian manifolds (Einstein, Sasakian, etc.)
53C26 Hyper-Kähler and quaternionic Kähler geometry, “special” geometry
53B35 Local differential geometry of Hermitian and Kählerian structures
53C15 General geometric structures on manifolds (almost complex, almost product structures, etc.)
53C35 Differential geometry of symmetric spaces

Keywords:

para-complex structure; para-Kähler structure; para-CR structure; para-Kähler Einstein manifold; semisimple Lie group; homogeneous manifold

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