Ford, Kevin
The distribution of integers with a divisor in a given interval. (English) Zbl 1181.11058
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This paper considers the number of integers $n \le x$ having a divisor in a given interval $(y, z]$, denoted by $H(x, y, z)$. The principal result gives the order of magnitude of $H(x, y, z)$ uniformly in all parameters, irrespective of the size of $z - y$ relative to x and y. Similarly one may write $H_r(x, y, z)$ for the number of integers $n \le x$ with exactly r divisors in $(y, z]$. For $r = 1$ the exact order of magnitude is found for any x, y, z with $z \le x^{1/2-\varepsilon}$, where ε is any fixed positive constant. When $r \ge 2$ is fixed the order of magnitude of $H_r(x, y, z)$ is also determined for large y and z in the range

$$y + \frac{y}{(\log y)^{\log 4-1-\varepsilon}} \le z \le \min\{y^C, x^{1/2-\varepsilon}\},$$

for any fixed $\varepsilon > 0$ and $C > 1$. These questions have previously been investigated by *R. R. Hall* [Sets of multiples, Cambridge Tracts in Mathematics. 118. Cambridge: Cambridge University Press (1996; Zbl 0871.11001)] and by *G. Tenenbaum* [(*) Compos. Math. 51, 243–263 (1984; Zbl 0541.10038) and Acta Arith. 49, No. 2, 165–187 (1987; Zbl 0636.10038)] amongst others. This second paper of Tenenbaum makes two conjectures on the relative sizes of $H(x, y, z)$ and $H_r(x, y, z)$. One of these is established in the present paper apart from one small range of the variables. The other conjecture states that if $r \ge 1$ and $\varepsilon > 0$ are fixed then for $y \le x^{1/2-\varepsilon}$ one has

$$H_r(x, y, z) = o(H(x, y, z))$$

as $z/y \to \infty$. The present paper proves this completely.

It is not hard to show that both $x^{-1}H(x, y, z)$ and $x^{-1}H_r(x, y, z)$ tend to limits as $x \to \infty$. These limits are denoted by $\varepsilon(y, z)$ and $\varepsilon_r(y, z)$, respectively. It was conjectured by *P. Erdős* [Vestn. Leningr. Univ. 15, No. 13 (Ser. Mat. Mekh. Astron. No. 3), 41–49 (1960; Zbl 0104.26804)] that $\varepsilon_1(y, 2y)/\varepsilon(y, 2y) \to 0$ as $y \to 0$. This is now shown to be false. More generally, one has

$$\frac{\varepsilon_r(y, \lambda y)}{\varepsilon(y, \lambda y)} \gg_{r,\lambda} 1$$

for any fixed $\lambda > 1$ and $r \in \mathbb{N}$.

The paper presents a number of corollaries of the main theorems. We mention just one. If $\rho(n)$ denotes the largest divisor d of n with $d \le \sqrt{n}$, then $\sum_{n \le x} \rho(n)$ has exact order

$$x^{3/2}(\log x)^{-\delta}(\log\log r)^{-3/2}.$$

The detailed estimates for $H_r(x, y, z)$ are quite complicated, but we shall describe here the results for $H(x, y, z)$, which will suffice to give the general flavour. Given $z > y \ge 4$ define η, u, β and ξ by $z = e^n y = y^{1+u}$, $\eta = (\log y)^{-\beta}$, and

$$\beta = \log 4 - 1 + \frac{\xi}{\sqrt{\log\log y}},$$

and set $\delta = 1 - \frac{1+\log\log 2}{\log 2}$ and

$$G(\beta) = \begin{cases} 1 + \frac{1+\beta}{\log 2}\log(\frac{1+\beta}{e\log 2}), & 0 \le \beta \le \log 4 - 1, \\ \beta, & \beta \ge \log 4 - 1. \end{cases}$$

In his paper (*), *G. Tenenbaum* showed that $H(x, y, z)$ has changes in behaviour around $z - y^2$, near

$z = 2y$, and in the vicinity of

$$z = z_0(y) := y \exp\{(\log y)^{1-\log 4}\}.$$

The main theorem of the paper then states that for $1 \leq y \leq y \leq x$ one has firstly four preliminary cases, namely

$H(x, y, z) = 0$ for $z < [y] + 1$;

$H(x, y, z) = [x([y] + 1)^{-1}]$ for $[y] + 1 \leq z < y + 1$;

$H(x, y, z)$ is of order 1 for $z > y + 1$ and $x \leq 100000$; and

$H(x, y, z)$ is of order x for $z > y + 1$, $y \leq \max(100, \sqrt{2})$ and $x > 100000$.

The main case is that in which $x > 100000$ and $100 \leq y \leq \min(z - 1, \sqrt{x})$, and here there are three subranges. When $y + 1 \leq z \leq z_0(y)$ the function $H(x, y, z)$ has order of magnitude ηx; for $z_0(y) \leq z \leq 2y$ it has order

$$\frac{x\beta}{\max(1, -\xi)(\log y)^{G(\beta)}};$$

and for $2y \leq z \leq y^2$ it has order of magnitude $u^\delta(\log 2/u)^{-3/2}$. Finally, when $x > 100000$, $\sqrt{x} < y < z \leq x$ and $z \geq y + 1$ the function $H(x, y, z)$ has the same order of magnitude as $H(x, x/z, x/y)$ if $x/y \geq x/z + 1$, and otherwise is of order ηx. In all these cases, when we say "$H(x, y, z)$ has order of magnitude $F(x, y, z)$", say, we mean that $F(x, y, z) \ll H(x, y, z) \ll F(x, y, z)$ with absolute implied constants.

Reviewer: Roger Heath-Brown (Oxford)

MSC:

11N25 Distribution of integers with specified multiplicative constraints

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divisors; interval; order of magnitude; Erdős conjecture

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