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On the Euler-Kronecker constants of global fields and primes with small norms. (English)

Zbl 1185.11069
Ginzburg, Victor (ed.), Algebraic geometry and number theory. In Honor of Vladimir Drinfeld's 50th birthday. Basel: Birkhäuser (ISBN 978-0-8176-4471-0/hbk). Progress in Mathematics 253, 407-451 (2006).

Let K be a global field, i.e., either an algebraic number field of finite degree, or an algebraic function field of one variable over a finite field. Let $\zeta_K(s)$ be the Dedekind zeta function of K, with the Laurent expansion at $s=1$:

$$\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + c_1(s-1) + \dots .$$

The author presents a systematic study of the real number

$$\gamma_K = \frac{c_0}{c_{-1}},$$

which he calls the Euler-Kronecker constant of K. Let

$$\Phi_K(x) = \frac{1}{x-1} \sum_{N(P)^k \le x} \left(\frac{x}{N(P)^k} - 1 \right) \log N(P) \quad (x>1),$$

where P is a (non-archimedean) prime divisor of K with the norm $N(P)$ and $k \ge 1$.

Suppose that K is a number field of degree $n = [K:\mathbb{Q}]$ and discriminant d_K. If the Generalized Riemann Hypothesis (GRH) is true then

$$\gamma_K < \left(\frac{\alpha_K+1}{\alpha_K-1} \right) (2\log\alpha_K + 1 - \Phi_K(\alpha_K^2)) \le \left(\frac{\alpha_K+1}{\alpha_K-1} \right) (2\log\alpha_K + 1),$$

provided that $n>2$, or $n=2$ and $|d_K|>8$. Here

$$\alpha_K = \frac{1}{2}\log|d_K|.$$

Let

$$\beta_K = -\frac{r_1}{2}(\gamma + \log 4\pi) - r_2(\gamma + \log 2\pi),$$

with r_1, r_2 the number of real and imaginary places of K, γ Euler's constant. Then, unconditionally,

$$\gamma_K > -\alpha_K - \beta_K - 1.$$

Let $n>1$ and

$$\alpha_K^* = \frac{\alpha_K}{n-1}.$$

If $\alpha_K^* > 1$ then, under GRH,

$$\frac{\alpha_K^*+1}{\alpha_K^*-1}(\gamma_K+1) > -2(n-1)(\log\alpha_K^* + 1).$$

Suppose that K is a function field of one variable with the constant field $\mathbb{F}_q$ and genus g. Then

$$\gamma_K < \left(\frac{\alpha_K+1}{\alpha_K-1} \right) (2\log\alpha_K + 1 + \log q - \Phi_K(\alpha_K^2)) \le$$

$$\le \left(\frac{\alpha_K+1}{\alpha_K-1} \right) (2\log\alpha_K + 1 + \log q).$$

Here

$$\alpha_K = (g-1)\log q.$$

It holds that

$$\gamma_K > -\alpha_K - \frac{q+1}{2(q-1)}\log q.$$

For fixed q:

$$\liminf \frac{\gamma_K}{(g_K-1)\log q} \geq -\frac{1}{\sqrt{q}+1}.$$

If K is an extension of $\mathbb{F}_q(t)$ of degree $n>1$ and

$$\alpha_K^* := \frac{(g-1)\log q}{n-1} > 1$$

then

$$\frac{\alpha_K^*+1}{\alpha_K^*-1}(\gamma_K + \frac{q+1}{2(q-1)}\log q) > -2(n-1)(\log\alpha_K^* + \frac{\alpha_K^*}{\alpha_K^*-1}).$$

For the entire collection see [Zbl 1113.00007].

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MSC:

11R42 Zeta functions and L-functions of number fields
11R47 Other analytic theory
11R58 Arithmetic theory of algebraic function fields

Keywords:

Euler-Kronecker constant; global field; Dedekind zeta function

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