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Discontinuous groups acting on homogeneous spaces of reductive type. (English)

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Kawazoe, T. (ed.) et al., Representation theory of Lie groups and Lie algebras. Proceedings of the conference, Fuji-Kawaguchiko, Japan, August 31–September 3, 1990. Hackensack, NJ: World Scientific (ISBN 981-02-1090-6). 59-75 (1992).

From the text: Let H be a closed subgroup of a Lie group G. The subject of this expository paper is roughly about the following:

Question A-0. How large a discrete subgroup of G can act properly discontinuously on a homogeneous space G/H?

Our concern will be mainly with the case where G/H is a homogeneous space of reductive type (Definition 5). If H is not compact, the action of a discrete subgroup Γ of G on G/H is not automatically properly discontinuous and the double coset $\Gamma\backslash G/H$ may be non-Hausdorff. This fact is the main difficulty in our problem. In fact, it may well happen that only finite subgroups of G can act properly discontinuously on G/H. For example, suppose that $G/H = \mathrm{SO}(n+1,1)/\mathrm{SO}(n,1)$, a pseudo-Riemannian homogeneous space of metric type $(n,1)$ and that Γ is a discrete subgroup of G. *E. Calabi* and *L. Markus* proved that $\Gamma\backslash G/H$ is Hausdorff if and only if Γ is a finite group [Ann. Math. (2) 75, 63–76 (1962; Zbl 0101.21804)]. Thus, a homogeneous space $G/H = \mathrm{SO}(n+1,1)/\mathrm{SO}(n,1)$ is somehow like a compact space. Named after their surprising discovery, such a homogeneous space is called to have a Calabi-Markus phenomenon.

In contrast to the above case with a noncompact isotropy subgroup H, *A. Borel* and *Harish-Chandra* [Ann. Math. (2) 75, 485–535 (1962; Zbl 0107.14804)] and *A. Borel* [Proc. Int. Congr. Math., Stockholm 1962, 10–22 (1963; Zbl 0134.16502)] showed that Riemannian symmetric spaces are rich in properly discontinuous actions. That is, let G be a real reductive linear Lie group and K a maximal compact group of G. Then there exists a discrete subgroup Γ of G such that the double coset space $\Gamma\backslash G/H$ is a compact (Hausdorff smooth) manifold. Also, there exists a discrete subgroup Γ such that the double coset space is noncompact manifold of finite volume.

On the other hand, even if the isotropy subgroup H is noncompact, it may also happen that a homogeneous space has a large discontinuous group Γ such that $\Gamma\backslash G/H$ is a compact manifold. Namely, this is an opposite extremal case to a Calabi-Markus phenomenon. A group manifold $G/H = G' \times G'/\operatorname{diag} G'$ is the case. We want to find other homogeneous spaces which admit large discontinuous groups.

For the entire collection see [Zbl 1098.22002].

MSC:

22E40 Discrete subgroups of Lie groups
22-02 Research exposition (monographs, survey articles) pertaining to topological groups