Let $X$ be a Calabi-Yau variety over $\mathbb{C}$. Let $G$ be a finite group acting on $X$, and consider the quotient $X/G$. This paper determines the place of $X/G$ in the birational classification of varieties. More precisely, the paper computes the Kodaira dimension of $X/G$ and determines when $X/G$ is uniruled or rationally connected. When $G$ has fixed points, the classification problem is answered by studying the stabilizer subgroups near the fixed points. These stabilizers are related to unitary reflection groups.

Let $V$ be a complex vector space and $g \in \mathrm{GL}(V)$ an element of finite order. Its eigenvalues can be written as $e(r_1), \ldots, e(r_n)$ where $e(x) := e^{2\pi i x}$ and $0 \leq r_i < 1$. Define the age of $g$ by $\text{age}(g) := r_1 + \cdots + r_n$.

Let $G$ be a finite group and $(\rho, V)$ a finite-dimensional complex representation of $G$. $\rho : G \to \mathrm{GL}(V)$ is said to satisfy the (local) Reid-Tai condition if $\text{age}(\rho(g)) \geq 1$ for every $g \in G$ for which $\rho(g)$ is not the identity.

Now let $G$ be a finite group acting on a smooth projective variety $X$. Then $G$-action satisfies the (global) Reid-Tai condition if for every $x \in X$, the stabilizer representation $\text{Stab}_x(G) \to \mathrm{GL}(T_xX)$ satisfies the (local) Reid-Tai condition.

**Theorem 1.** Let $X$ be a uniruled projective Calabi-Yau variety and $G$ a finite group acting on $X$. Then the following are equivalent:

1. $\kappa(X) = 0$.
2. $X/G$ is not uniruled.
3. The $G$-action satisfies the global Reid-Tai condition.

It can happen that $X/G$ is uniruled, but not rationally connected. The following result gives characterizations of uniruled and rationally connected $X/G$.

**Corollary 4.** Let $X$ be a smooth, simply connected projective Calabi-Yau variety that is not a nontrivial product of two Calabi-Yau varieties. Let $G$ be a finite group acting on $X$. Then the following are equivalent:

1. $X/G$ is uniruled.
2. $X/G$ is rationally connected.
3. The $G$-action does not satisfy the global Reid-Tai condition.

The next result is the classification of representations that satisfy local Reid-Tai condition. The groups that have some representations violating the Reid-Tai condition are closely related to complex reflection groups.

Let $G$ be a finite group and $(\rho, V)$ a finite-dimensional complex representation of $G$ such that $(\rho, V)$ does not satisfy the (local) Reid-Tai condition. So there is an element $g \in G$ such that $0 < \text{age}(\rho(g)) < 1$. Such a pair $(G, V)$ is called a non-RT pair, and $g$ an exceptional element. Further, $(G, V)$ is called a basic non-RT pair if the conjugacy class of any exceptional element $g \in G$ generates $G$. A basic non-RT pair is said to be of reflection type if $(G, V)$ is projectively equivalent to some reflection group $(G', V')$. It is known that every reflection group is a non-RT pair. The classification result is obtained for basic non-RT pairs up to projective equivalence.

**Theorem.** Up to projective equivalence, there are only finitely many basic non-Reid-Tai pairs that are not reflection type.

For the entire collection see [Zbl 1185.00042].

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MSC:
14J32 Calabi-Yau manifolds (algebro-geometric aspects)
14K05 Algebraic theory of abelian varieties
20E99 Structure and classification of infinite or finite groups
14M20 Rational and unirational varieties
14E05 Rational and birational maps
20F55 Reflection and Coxeter groups (group-theoretic aspects)

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