Given a smooth $n$-manifold $M$ and a proper subring $\Lambda$ of $\mathbb{R}$, the smooth cohomology group $\hat{H}^{k+1}(M;\Lambda)$ is the group of homomorphisms $\chi$ from the group of smooth singular $k$-cycles $Z_k(M;\mathbb{R})$ into $\mathbb{R}/\Lambda$ with the property that for every smooth $(k+1)$-form $\omega \in \Omega^{k+1}(M)$ such that $\chi(\partial \omega) = \int_M \omega + \Lambda$ for every smooth singular $(k+1)$-chain $c \in C_{k+1}(M;\mathbb{R})$. The relations with usual cohomology are as follows.

A nonvanishing real form never takes values in a proper subring of $\mathbb{R}$; consequently $\omega$ must be closed, and $R : \hat{H}^k(M;\Lambda) \to \Omega^k(M) = \ker d\Omega^k(M)$, $\chi \mapsto \omega$, is well-defined. It turns out that $\im R$ equals the kernel of the composition $\Omega^k_{cl}(M) \to H^k(M;\mathbb{R}) \to H^k(M;\mathbb{R}/\mathbb{Z})$, and $\ker R$ is isomorphic to $H^{k-1}(M;\mathbb{R}/\mathbb{Z})$. On the other hand, since $\mathbb{R}$ is divisible, every $\omega$ as above lifts to a homomorphism $\hat{Z}_k(M;\mathbb{R}) \to \mathbb{R}$, which in turn is the restriction of a real singular cochain $\phi \in C^k(M;\mathbb{R})$. Since $\omega$ is closed and well-defined, the cochain $\psi \in C^{k+1}(M;\Lambda)$, $\psi(c) = \delta \phi(c) - \int_M \omega$, is a cocycle, and $I : \hat{H}^{k+1}(M;\Lambda) \to H^{k+1}(M;\Lambda)$, $\chi \mapsto [\psi]$, is well-defined.

It turns out that $I$ is an epimorphism, and its kernel is isomorphic to the cokernel of the composition $H^k(M;\Lambda) @ \to c >> H^k(M;\mathbb{R})\Omega^k(M)/\im d [\text{J. Cheeger and J. Simons, Geometry and topology, Proc. Spec. Year, College Park/Md. 1983/84, Lect. Notes Math. 1167, 50-80 (1985; Zbl 0621.57010}]. A geometric description of $\hat{H}^k(M;\mathbb{Z})$ in the spirit of S. Buoncristiano, C. P. Rourke and B. J. Sanderson [A geometric approach to homology theory. London Mathematical Society Lecture Note Series. 18. Cambridge etc.: Cambridge University Press. (1976; Zbl 0315.55002)] has been given by the authors and M. Kreck [Ann. Math. Blaise Pascal 17, No. 1, 1–16 (2010; Zbl 1200.55007)].

Now let $h$ be a generalized (in the usual sense) cohomology theory, and let us fix a natural transformation of theories $\epsilon : h^*(X) \to H(X;V)^*$, where $V$ is a $\mathbb{Z}$-graded vector space and $H(X;V)^* = \sum H^i(X;V^{k-i})$. The principal example is as follows: $V = h^*(pt) \otimes \mathbb{R}$ and $c$ is the real-valued Chern–Dold character $h^*(X) @ \to c h_k @ >> H(X;h^*(pt) \otimes \mathbb{Q})^* @ >> i^* @ >> H(X;h^*(pt) \otimes \mathbb{R})^*$, that is the unique natural transformation of theories that for $X = pt$ is given by $a \mapsto a \otimes 1$. (Apart from A. Dold’s original paper [Colloq. algeb. Topology, Aarhus 1962, 2–9 (1962; Zbl 0145.20104)], a brief construction of $\epsilon_k$ appears in the Hopkins–Singer paper cited below, and a more detailed treatment is found in V. M. Buchstaber’s survey [J. Sov. Math. 11, 815–921 (1979; Zbl 0428.55002)].) The authors define a “smooth extension” of $h$, which is a functor $\tilde{h}$ from the category of smooth manifolds and smooth maps to $\mathbb{Z}$-graded abelian groups, along with additional data:

(i) an epimorphic natural transformation $I \tilde{h}(\cdot) \to h^*(\cdot)$;

(ii) a natural equivalence $a$ of degree $-1$ between ker $I$ and the cokernel of the composition $h^*(\cdot) @ >> c >> H(M;V)^*[\Omega(M)/\im d]^* \otimes \mathbb{R} V$; and

(iii) a natural transformation $R : \hat{h}(\cdot) \to \Omega^1_{cl}(\cdot)^* \otimes \mathbb{R} V$ such that $R a^{-1} = d$ and $c l = q R$, where $q$ is the surjection $\Omega^1_{cl}(\cdot)^* \otimes \mathbb{R} V \to H(M;V)^*$.


As a motivation for considering smooth extensions of generalized cohomology theories the present authors mention “the problem of setting up Lagrangians for quantum field theories with differential form field strength”.

In the paper under review, the authors use geometric structures on vector bundles and submersions, and analytic methods, to construct a smooth extension $\hat{K}$ of complex $K$-theory with respect to the real-valued Chern character $i^* c_K$, along with a natural pushforward $\hat{p} : \hat{K}(E) \to \hat{K}(B)$ for a smoothly $K$-oriented proper submersion $p : E \to B$, satisfying a number of properties. They also construct a lifted Chern character $\tilde{c}_K : \hat{K}^*(M) \to \hat{H}(M;\mathbb{Q})^*$ and show that it induces a rational isomorphism.
Moreover, they construct a modified pushforward $\hat{\mu}^!_A : \hat{H}(E; \mathbb{Q})^* \to \hat{H}(B; \mathbb{Q})^*$ and prove a smooth extension of the cohomological Atiyah-Singer index theorem: $\hat{\mu}^!_A \hat{c}_K = \hat{c}_K \hat{\mu}^!$.

The question of uniqueness of smooth extensions of generalized cohomology theories has been studied previously by the authors [U. Bunke and T. Schick, J. Topol. 3, No. 1, 110–156 (2010; Zbl 1252.55002); correction, arXiv:1007.2788]. In particular, they proved that complex $K$-theory admits infinitely many inequivalent smooth extensions with respect to $i^* c_K$, but at most one such extension $\hat{K}$ admitting a natural pushforward $\hat{\mu} : \hat{K}(M \times S^1) \to \hat{K}(M)$ for the projection $p M \times S^1 \to M$, where $M$ is an arbitrary smooth manifold, satisfying $\hat{\mu} p^* = 0$ and compatible with $a, I$ and $R$. The extension $\hat{K}$ and the pushforward $\hat{\mu}$ constructed in the paper under review do satisfy these properties.

The “flat” part of a smooth extension $\hat{\mu}$ of a pair $(h, c)$, that is the kernel of the “curvature” transformation $R$, is a homotopy invariant functor; the flat part of the Hopkins-Singer $\hat{\mu}$ is more specifically the restriction to smooth manifolds of a generalized (in the usual sense) cohomology theory $h_{R/Z}$, “the homotopy theorist’s version of $h$ with $\mathbb{R}/\mathbb{Z}$ coefficients” [the authors, J. Topology, op. cit.]. (See Buoncristiano-Rourke-Sanderson, op. cit., for a geometric topologist’s definition of $h_{R/Z}$.)

In the present paper, the authors conclude from their previous results that the even-graded part and the flat part of their $\hat{K}$ coincide with those of the Hopkins-Singer smooth extension of complex $K$-theory. On the flat part $K_{R/Z}$ of $\hat{K}$, which is further identified with the “multiplicative $K$-theory” $\text{Hom}(K_*(M), \mathbb{R}/\mathbb{Z})$, the authors’ pushforward $\hat{\mu}$ and lifted Chern character $\hat{c}_K$ coincide with those studied previously by J. Lott.

For the entire collection see [Zbl 1192.00075].

Reviewer: Sergey Melikhov (Moskva)

MSC:

19L10 Riemann-Roch theorems, Chern characters
58J28 Eta-invariants, Chern-Simons invariants
55N20 Generalized (extraordinary) homology and cohomology theories in algebraic topology
57R19 Algebraic topology on manifolds and differential topology

Keywords:

Deligne cohomology; smooth $K$-theory; Chern character; families of elliptic operators; Atiyah-Singer index theorem

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