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Rational approximations of values of the Gamma function on rationals. (Approximations rationnelles des valeurs de la fonction Gamma aux rationnels.) (French) Zbl 1206.11095

J. Number Theory 130, No. 4, 944-955 (2010).

Let $x > 0$ be a real number and α a complex number with real part > -1. The author produces explicit linear recurrences of order 3

$$C_3 u_{n+3} + C_2 u_{n+2} + C_1 u_{n+1} + C_0 u_n = 0,$$

with coefficients C_0, C_1, C_2, C_3 which are polynomial in n, α and x of total degrees ≤ 16, having solutions $(P_n(x,\alpha))_{n\geq 0}$ and $(Q_n(x,\alpha))_{n\geq 0}$, sequences of polynomials in $\mathbb{Q}[\alpha, x]$, for which the sequence $\big(P_n(x,\alpha)/Q_n(x,\alpha)\big)_{n\geq 0}$ converges quickly towards $\Gamma(1+\alpha)/x^\alpha$. In the special case $x = 1$ and $1+\alpha = a/b \in \mathbb{Q}_{>0}$, this yields new rational approximations to $\Gamma(a/b)$.

The proof rests on the methods of the previous work by the author [Trans. Am. Math. Soc. 361, No. 11, 6115–6149 (2009; Zbl 1236.11061)] involving the sequence of polynomials

$$A_{n,\alpha}(x) = \frac{1}{n!^2} e^x \big(x^{n-\alpha}(e^{-x}x^{n+\alpha})^{(n)}\big)^{(n)}$$

previously introduced by *A. I. Aptekarev, A. Branquinho* and *W. Van Assche* [Trans. Am. Math. Soc. 355, No. 10, 3887–3914 (2003; Zbl 1033.33002)]. These polynomials $A_{n,\alpha}$ are denominators of Padé simultaneous approximants at infinity to the functions $\mathcal{F}_0$ and $\mathcal{F}_\alpha$, where, for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $\alpha > -1$,

$$\mathcal{F}_\alpha(z) = \int_0^\infty \frac{t^\alpha e^{-t}}{z-t} dt.$$

Reviewer: Michel Waldschmidt (Paris)

MSC:

11J91	Transcendence theory of other special functions
33B15	Gamma, beta and polygamma functions

Keywords:

Euler Gamma function; Rational Diophantine approximation; Ternary linear recurrence sequences; Padé approximation

Software:

MultInt

Full Text: DOI

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