Görtz, Ulrich; Wedhorn, Torsten

Algebraic geometry I. Schemes. With examples and exercises. (English) Zbl 1213.14001

Advanced Lectures in Mathematics. Wiesbaden: Vieweg+Teubner (ISBN 978-3-8348-0676-5/pbk). vii, 615 p. (2010).

The volume at hand is the first part of a profound introduction to algebraic geometry in its modern setting, that is, in A. Grothendieck's revolutionary conceptual framework of algebraic schemes. In these days, about fifty years after Grothendieck's refoundation of algebraic geometry in the language of schemes, this approach is well-established as both a fundamental cornerstone and an indispensable toolkit for the current research in various fields of mathematics, including geometry, number theory, complex analysis, theoretical physics, and their many modern applications.

As the authors point out in the preface, the present textbook is to give a systematic and comprehensive introduction to the theory of schemes in its necessary generality, thereby providing the reader with a solid background for turning towards current research in algebraic geometry and its related areas within contemporary mathematics.

Actually, there is already a considerably large number of excellent textbooks on modern algebraic geometry explaining some basics of scheme theory as well, at least so to a reasonable extent. However, most of the great standard texts have their own special focus, use schemes as an appropriate, well-adapted language for certain parts of the exposition, and develop their general theory only as far as needed for particular purposes. In fact, the monumental pioneering work "Elements of Algebraic Geometry" (EGA I–IV) by A. Grothendieck and J. Dieudonné is still the only encyclopedic reference for scheme theory, but this multi-volume, highly abstract treatise is very far from being a suitable textbook for beginners in the field.

On the other hand, the volume under review is really written as a comprehensive textbook of introductory character, almost exclusively devoted to the theory of schemes, primarily geared toward graduate students, and thereby assuming only basic knowledge in abstract algebra and topology as prerequisites. Moreover, the authors have set a high value on carefully thought out didactic principles underlying the entire exposition, and that by developing the abstract, both conceptually and technically utmost demanding theory of schemes all through in very systematic, detailed, motivating, concrete and illustrating a manner.

As to the contents, the present volume comprises sixteen chapters and five supplementing appendices.

The introduction to the book briefly explains the main concern of both classical algebraic geometry and the theory of schemes, on the one hand, and provides then a practical guide for the use of the text, including some directing comments on the contents of the single chapters, on the other hand.

Chapter 1 begins with a discussion of affine algebraic varieties, prevarieties, and projective varieties as spaces with (algebraic) functions, that is, as both historical precursors and first important examples related to abstract algebraic schemes à la Grothendieck. Chapter 2 introduces the spectrum of a ring, its Zariski topology, the basics of sheaf theory, and finally affine schemes as locally ringed spaces forming the local building blocs for general schemes treated in the sequel.

Chapter 3 introduces the main objects of study of the book, namely schemes and their morphisms. Basic properties of schemes and morphisms in general, projective schemes, schemes associated to prevarieties, subschemes, and immersions of schemes are the principal topics discussed in this chapter.

The category of schemes and functors attached to schemes are more closely investigated in Chapter 4, with the focus on fiber products, base change properties, and the structure of fibers of a morphism.

In this context, other fundamental constructions such as inverse images and schematic intersections of subschemes, morphisms and products of projective spaces, the Segre embedding, and group schemes are treated along the way.

After the basics of scheme theory (as developed in Chapters 2–4), the more advanced part of the whole subject starts with Chapter 5, where schemes over a general ground field are studied, including the notion of dimension of a scheme of finite type over a field and, as a first application of the general theory developed so far, the intersection calculus for plane curves.

Chapter 6 is concerned with local properties of schemes, with particular emphasis on algebraic tangent spaces, smooth morphisms, regular schemes, and normal schemes.

Quasi-coherent module sheaves, their basic properties, and the fundamental constructions of these objects are described in Chapter 7, whereas the functorial viewpoint in scheme theory is further pursued in the subsequent Chapter 8. The latter part deals with representable functors in general, then with representable morphisms of schemes, Zariski sheaves, Zariski coverings of functors, and two important concrete examples: Grassmannians and Brauer-Severi schemes.

Chapter 9 turns to the notions of separated schemes and separated morphisms in its first part, while the second part is dedicated to the important topic of rational maps and function fields of schemes, thereby providing the first steps into birational geometry.

Generalizing the notion and properties of schemes locally of finite type over a field, Chapter 10 analyzes various finiteness conditions for arbitrary morphisms of schemes, constructible properties of schemes and quasi-coherent sheaves as well as the structure of schemes over inductive limits of rings. This material provides fundamental concepts and techniques for the further general study of schemes, and the basic reference for it is the comprehensive volume "EGA IV" by Grothendieck and Dieudonné from the 1960s.

Chapter 11 treats two other central topics in algebraic geometry, namely vector bundles on schemes and divisors. This is done for general schemes (over a base scheme), with particular emphasis on locally free sheaves, line bundles, Cartier divisors, Weil divisors. Picard groups, divisor class groups, and the relations between these objects. Along the way, flattening stratifications for quasi-coherent modules of finite type, torsors, and non-Abelian cohomology are also briefly touched upon.

The main objects studied in Chapter 12 are affine, finite and proper morphisms. Apart from Chevalley's and Serres criteria for characterizing affine morphisms, the concept of normalization of an integral scheme and Zariski's Main Theorem appear in the limelight of this section of the book. Next, in Chapter 13, projective schemes are studied in greater detail, together with their distinguished quasi-coherent module sheaves and their embeddings into projective space. This includes the discussion of ample and very ample line bundles, immersions into projective bundles, linear systems, (quasi-)projective morphisms, the study of blow-ups, a general version of Chow's Lemma, and an outlook to the resolution of singularities likewise. Chapter 14 describes flat morphisms, their important geometric properties, and their various characterizations, in particular the valuative criterion for flatness.

In this context, a large number of results obtained from the principle of faithfully flat descent for schemes, quasi-coherent modules, and torsors is derived, together with a concrete application to Brauer-Severi varieties (as introduced in Chapter 8). The next parts are then dedicated to a refined treatment of dimension theory, especially with regard to the variation of the dimension of the fibers of a flat morphism, on the one hand, and with respect to Cohen-Macaulay schemes on the other. This chapter ends with a first short glimpse of the idea of Hilbert schemes as parametrizing objects. Chapter 15 gives an application of the general theory developed so far to the important class of Noetherian schemes of dimension one, i.e., to absolute algebraic curves and their divisors, ending with an outlook to the Riemann-Roch Theorem for curves. Finally, several important classes of concrete schemes are exhibited in the concluding Chapter 16. A special didactic feature of the presentation given here is that these examples are discussed in parallel to the advancement of the theory in the main part of the book, with respective references to the related previous chapters. More precisely, determinantal varieties and schemes, the Clebsch cubic surface and its relation to a special Hilbert modular surface, some quotients of algebraic surfaces by cyclic groups, and Abelian varieties are used as examples to illustrate many of the concepts, methods, and results developed in the single chapters of the book, and that in very vivid and instructive a manner.

In addition, there are five appendices at the end of the book.

Appendix A recalls some basic notions and result from category theory, whereas Appendix B provides a collection of those relevant facts from commutative algebra as they are used in the course of the main text. Appendix G contains a list of properties of morphisms of schemes satisfying various permanence principles, and an overview of the many relations between different properties of morphisms of schemes is given in Appendix D. Finally, the authors recall (and partly refine) several definitions and properties concerning constructible subsets of schemes in Appendix E. Generally, these appendices have been added for the convenience of the reader, basically in order to keep the book as self-contained as possible, on the one hand, and to increase both its lucidity and its utility as a reference book on the other.

Another outstanding feature of the present primer of basic scheme theory is given by the exceptionally large number of illustrating examples, supplementing remarks, and accompanying exercises permeating the entire text. In fact, each chapter concludes with a set of about thirty related, carefully selected working problems, the solutions of which are to provide some important enrichment, refinement, and completion of the respective main text, apart from serving as a highly valuable testing ground for the reader's understanding of the core material.

Finally, the reader finds a rich bibliography of related textbooks, a detailed list of contents of the single chapters, an extensive index of symbols used in the course of the text, and a sweeping, virtually complete subject index at the end of the book.

To sum up, the book under review furnishes an excellent introduction to the basic theory of algebraic schemes and their morphisms. While presenting the highly abstract and technically utmost demanding material in full generality, and therefore in its widest applicability, the authors have consistently striven to keep it accessible even for beginners in the field. In regard to its comprehensiveness, lucidity, depth, versatility and didactical conception, this book is rather unique within the relevant textbook literature, and therefore an utmost valuable replenishment of the latter. Moreover, the authors have produced the so far most profound introduction to the original EGA volumes by Grothendieck and Dieudonné, which certainly must be seen as another rewarding result of their overall work.

As the title of the current book suggests, there will be a forthcoming second volume, with focus on the cohomology of schemes and its applications. To doubt, these two volumes together will represent another significant standard text in contemporary algebraic geometry and its allied areas within mathematics as a whole.

Reviewer: Werner Kleinert (Berlin)

MSC:

14-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to algebraic geometry
14A15 Schemes and morphisms
14C20 Divisors, linear systems, invertible sheaves
14F05 Sheaves, derived categories of sheaves, etc. (MSC2010)
14L15 Group schemes
14M12 Determinantal varieties
14L30 Group actions on varieties or schemes (quotients)
14B05 Singularities in algebraic geometry

Keywords:

textbook (algebraic geometry); schemes and morphisms; prevarieties; quasi-coherent sheaves; vector bundles; divisors; algebraic curves; determinantal varieties; singularities