Let $\Gamma$ be an infinite, finitely generated, residually finite group. The theme of the paper under review is to explore under which circumstances $\Gamma$ admits a family of finite index subgroups satisfying certain sequential properties, motivated by Bertrand’s postulate. The investigation is linked to studying two numerical invariants of the group $\Gamma$, the residual average $\text{Ave}(\Gamma)$ and the normal residual average $\text{Ave}^\Sigma(\Gamma)$. These invariants lie in the real interval $(2, \infty)$ and their precise values depend, in an intricate way, on the growth rates of indices of finite index subgroups in $\Gamma$. The main result is that, if $\Gamma$ admits a finite dimensional linear representation in characteristic 0, then the invariants $\text{Ave}(\Gamma)$ and $\text{Ave}^\Sigma(\Gamma)$ are finite.

For a compact topological group $G$ let $S(G)$ denote the collection of all open subgroups of $G$. The divisibility function on such a group $G$ is the function

$$D_G : G \to \mathbb{N} \cup \{\infty\}, \quad g \mapsto \inf\{|G : H| \mid g \not\in H \in S(G)\}.$$ 

With these preliminaries in place, the residual average of a residually finite group $\Gamma$ is defined to be

$$\text{Ave}(\Gamma) = \int_{\hat{\Gamma}} D_{\hat{\Gamma}}(x) \, d\mu(x),$$ 

where $\hat{\Gamma}$ denotes the profinite completion of $\Gamma$ and the Haar measure $\mu$ on $\hat{\Gamma}$ is normalised so that $\mu(\hat{\Gamma}) = 1$. The normal divisibility function $D^\Sigma_\Gamma$ and the normal residual average $\text{Ave}^\Sigma(\Gamma)$ are defined in a similar way, by restricting attention to normal open subgroups.

Theorems 1.3 and 1.4 of the paper state that, if $\Gamma$ is an infinite, finitely generated, linear group over $\mathbb{C}$, then $\text{Ave}(\Gamma)$ and $\text{Ave}^\Sigma(\Gamma)$ are finite. The proof relies on the following theorem. For every infinite, finitely generated, linear group $\Gamma$ over $\mathbb{C}$ there exist a finite index subgroup $\Gamma_0$, a positive constant $d$ and a ‘prime’ family $\Delta_j$, $j \in \mathbb{N}$, of normal finite index subgroups of $\Gamma_0$ such that

$$|\Gamma : \Delta_j| < |\Gamma : \Delta_{j+1}| \leq d |\Gamma : \Delta_j|.$$ 

To say that the family $\Delta_j$, $j \in \mathbb{N}$, is prime means that $\Gamma_0 = \bigcap_{i=1}^n \Delta_i$ for all $j \in \mathbb{N}$. The theorem is motivated by the following special case: if $\Gamma = \Gamma_0 = \mathbb{Z}$ and $\Delta_j = p_j \mathbb{Z}$, where $p_j$ denotes the $j$th prime number, then by Bertrand’s postulate the desired inequality for indices holds for $d = 2$. The proof of the general result depends on the ‘Lubotzky alternative’ and the Strong Approximation Theorem for arithmetic groups. The authors also show how a weaker version of their result can be proved without using these tools.

The paper further contains two examples of groups with infinite normal average. The first Grigorchuk group, which is a subgroup of the automorphism group of an infinite rooted binary tree, provides an example of a finitely generated, residually finite group which is non-linear and has infinite normal average. The compact $p$-adic Lie group $\text{SL}_n(\mathbb{Z}_p)$ is given as an example of a linear group, which is not finitely generated (as an abstract group) and has infinite normal average. The paper concludes with a few remarks on averaging over other kinds of densities and on connections to zeta functions which are defined in terms of indices of normal subgroups.

The paper is very readable and well motivated, but unfortunately it contains several minor slips. For instance, claims made about the normal subgroup structure of the group $\text{SL}_n(\mathbb{Z}_p)$ are not completely correct when $p \mid n$.

Reviewer: Benjamin Klopsch (Egham)

MSC:

20E07 Subgroup theorems; subgroup growth
20E18 Limits, profinite groups
20E26 Residual properties and generalizations; residually finite groups
Keywords:
Bertrand's postulate; divisibility function; residual average; subgroup growth; residual finiteness; subgroups of finite index

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