Lebl, Jiří; Lichtblau, Daniel
Uniqueness of certain polynomials constant on a line. (English) Zbl 1229.13024
Linear Algebra Appl. 433, No. 4, 824-837 (2010).

Let $\mathcal{H}(2,d)$ denote the set of polynomials $p(x,y)$ of degree $d$ with nonnegative coefficients such that $p(x,y) = 1$ whenever $x+y = 1$. Let $N = N(p)$ denote the number of distinct monomials for a given $p$ in $\mathcal{H}(2,d)$. J. D’Angelo, S. Kos and E. Riehl [J. Geom. Anal. 13, No. 4, 581–593 (2003; Zbl 1052.26016)] showed that $d \leq 2N - 3$. Further, this inequality is sharp. For each odd $d$, let $f_d$ in $\mathcal{H}(2,d)$ be defined as

$$f_d(x,y) := \left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^d + \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^d + (-1)^{d+1}y^d.$$  

It is a fact that $d = 2N(f_d) - 3$. These family of polynomials have many other interesting properties. In the context of CR geometry, the CR maps that arise from $f_d$ are one of the only two possible classes of group invariant maps of balls. The polynomials $f_d$ are also related to Chebyshev polynomials, arise in denesting radicals, and have connections to number theory, for example, $f_d(x,y) = x^d + y^d \pmod{d}$ if and only if $d$ is an odd prime.

If $p \in \mathcal{H}(2,d)$ minimizes $N(p)$ for a fixed $d$, then $p$ is called a sharp polynomial. It can be proved that for a fixed degree only finitely many sharp polynomials exist. It is natural to attempt a classification of all sharp polynomials of a given degree. A related problem consists in finding sufficient and necessary conditions for $f_d$ to be the unique sharp polynomial in $\mathcal{H}(2,d)$ up to swapping of variables.

Using computational methods, the authors give a complete classification of sharp polynomials for degrees up to $d = 17$ extending previous work on the topic. Along the way, the authors provide new theoretical results about the form of sharp polynomials that are of independent interest. The authors describe two different methods to find sharp polynomials; first by computing the nullspace of certain matrices and second by translating the problem into a mixed-integer programming problem. Finally, the authors also describe degrees in which uniqueness definitely fails.

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MSC:

13P05 Polynomials, factorization in commutative rings
05D10 Ramsey theory
32H02 Holomorphic mappings, (holomorphic) embeddings and related questions in several complex variables
32H35 Proper holomorphic mappings, finiteness theorems
90C11 Mixed integer programming

Keywords:
CR geometry; proper map; degree bounds; real algebraic geometry; mixed linear programming

Software:
gmp; Mathematica; OEIS; Genius

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References: