A marked lattice polygon $(Q, A)$ with just one interior point gives rise to a certain toric surface $X_A$. Then, a general polynomial $f \in \mathbb{K}[x, y]$ having $A$ as support gives rise to an elliptic curve $C_f$ on the surface $X_A$, and one can compute the $j$–invariant of $C_f$.

Suppose the field $\mathbb{K}$ is endowed with a real valued non–archimedean valuation. Then one can compute $\text{Trop}(C_f)$, the tropicalization of $C_f$ and, under certain hypothesis, this is a genus one graph. The properties satisfied by the lattice length of the cycle contained in this graph remind us of the classical $j$–invariant, as has been observed by Vigeland, Mikhalkin and Kerber and Markwig.

The question arises: What is the relationship between these two numbers? In this paper, the authors provide an answer by showing that the valuation of the $j$–invariant of $C_f$ equals the negative of the lattice length of the cycle in $\text{Trop}(C_f)$. This is the main theorem. Previous work by the same authors in 2008 [J. Algebra 320, No. 10, 3832–3848 (2008; Zbl 1185.14030)] showed this result only for plane projective smooth cubics. For a general toric variety $X$, and under the hypothesis that $f$ induces a triangulation on the Newton polygon $N(f)$ all whose triangles are of minimal area, a similar result has been proved by D. E. Speyer “Uniformizing tropical curves I: genus zero and one,” arXiv:0711.2677.

In the paper, the authors define the tropical $j$–invariant and give a formula to compute it. They also prove that this number does not change under integral unimodular affine transformations. Then, the proof of the main theorem reduces to checking the statement for the 16 cases of marked lattice polygons with just one interior point. These, in turn, reduce to only 3 basic cases. The proof for these 3 basic cases requires certain procedures which use the following computer algebra programs: SINGULAR, POLYMAKE and TOPCOM. The last section of the paper explains in detail how these programs are used to the purpose of the proof. There are four appendices to this paper, which can be obtained online. The first appendix, invariant.lib, is a SINGULAR library containing most of the procedures described in the paper. The other three appendices contain the outputs and formulae computed for the 3 basic cases.

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MSC:

- 14T05 Tropical geometry (MSC2010)
- 14H52 Elliptic curves
- 51M20 Polyhedra and polytopes; regular figures, division of spaces

Keywords:

tropical geometry; elliptic curves; $j$-invariant; SINGULAR; POLYMAKE; TOPCOM

Software:

tropical.lib; invariant.lib; polymake; TOPCOM; SINGULAR

Full Text: DOI arXiv

References:
