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An introduction to measure theory. (English) Zbl 1231.28001

Graduate Studies in Mathematics 126. Providence, RI: American Mathematical Society (AMS) (ISBN 978-0-8218-6919-2/hbk). xvi, 206 p. (2011).

The book is primarily concerned with measure theory in $\mathbb{R}^n$ and integration of real-valued and complex-valued functions. It requires only moderate prior knowledge of real analysis, as can be expected after the completion of rigorous undergraduate courses covering differentiation and Riemann integration. Beside that prior knowledge, some proficiency with manipulation of sets is needed, although aside from a few exercises where it is required, no knowledge of ordinals or cardinals is assumed. Oddly, the book's 200 pages are divided into only two chapters, which I will treat here separately.

Contents of chapter 1. The first three sections are concerned with a motivation for and the construction of the Lebesgue measure and Lebesgue integration. The presentation is very concrete, starting with elementary sets and Jordan measure, explaining their shortcomings and stressing the advantages of the more sophisticated Lebesgue measure. Section 3 culminates in an in-depth discussion of Littlewood's three principles, Egorov's theorem, and Lusin's theorem. Section 4 introduces abstract measure spaces and their integration theory. The presentation strongly follows the form and order of the presentation of the first three sections and builds quite heavily on the intuition developed in them. The section contains the monotone convergence theorem, the Borel-Cantelli lemma, Fatou's lemma, and the dominated convergence theorem as well as many variations and enhancements of these results. Section 5 introduces the various modes of convergence and studies the intricate interrelations between them at some length. Section 6, concerned with differentiation theorems, starts out by recalling basic facts of classical differentiation theory, including the proofs (e.g., Rolle's theorem, the fundamental theorems etc.). The first goal is to establish the Lebesgue differentiation theorem using the one-sided Hardy-Littlewood maximal inequality. It is here that the reader is first introduced, with great care and patience, to the density argument, which recurs later. The next aim is to establish the Lebesgue differentiation theorem in general dimension. The third main goal of Section 6 is to establish the second fundamental theorem for absolutely continuous functions. Functions of bounded variation are discussed as well. Among the numerous exercises one finds the Steinhaus theorem, a construction of a Weierstrass function and Cantor's function, and Cousin's theorem. Section 7, which closes chapter 1, culminates in the Fubini-Tonelli theorem. Carathéodory's extension theorem is presented in full generality and is used to introduce Lebesgue-Stieltjes measures. It is then applied to product spaces to establish, in detail, the culminating result.

Contents of chapter 2. Section 1 is a list of general tips for solving problems in measure theory (and generally in mathematics). Each note is elaborated and includes several references to places in the text where it can be used effectively. This section can be read either before, at any point during, or after reading chapter 1 with almost certainly something to be gained with each reading. Section 2 is then a proof of the Rademacher differentiation theorem stressing the use of Fubini's theorem in carrying results from dimension one to higher dimensions. Section 3 very briefly discusses the terminological adaptation used in probability spaces and Section 4, again briefly, discusses Kolmogorov extension and hints at the usefulness of it in probability theory.

About the style of the book. The approach to measure theory taken is very concrete. Abstract measure spaces are only introduced after the reader had seen non-trivial results of Lebesgue measure and integration, and throughout the book there is a strong emphasis on the geometric flavour of the theory. The presentation in general is that of an evolving story with many remarks and elaborations that explain the why and how behind theorems, striving to foster in the reader an understanding of the big picture. Great care is taken to provide readable proofs and many exercises are scattered throughout the text.

A note to the student. The book is well suited for self-study, particularly for students interested in a very concrete approach to the subject. The numerous exercises, which evidently form an integral part of the book, will more than suffice, if worked out, to create mastery of the core concepts and techniques.

A note to the lecturer. The book must be followed pretty much in a linear order to truly benefit from its carefully crafted form. This means that it offers somewhat restricted flexibility in case it is to be used as a textbook for an introductory course on measure theory. However, if a course is to be designed to closely follow the book then the many exercises it contains offer great flexibility, allowing the lecturer to assign more exercises as homework or workout exercises herself to adapt the level of difficulty of the course.

Reviewer: Ittay Weiss (Suva)

MSC:

28-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to measure and integration

28A20 Measurable and nonmeasurable functions, sequences of measurable functions, modes of convergence

28A25 Integration with respect to measures and other set functions

28A35 Measures and integrals in product spaces

Keywords:

Measure theory