“Invitation to Classical Analysis” is not meant to be a usual introduction to calculus or real analysis. In fact, this book is designed for undergraduate students who are already familiar with the basic principles of real analysis, as taught in advanced calculus courses, and are prepared to explore a number of additional, highly substantial and utmost beautiful topics in classical analysis. Among those topics are Fourier series and integrals, classical inequalities, infinite products, approximation theory, the gamma function, Bernoulli numbers and polynomials, the Riemann zeta function, Tauberian theorems, elliptic integrals, ordinary differential equations and their power series solutions, special functions, space-filling curves, and others, most of which do not occur any more in the modern undergraduate curriculum as natural sequels to the abstract theory conveyed in contemporary advanced calculus courses. As the author puts it, the purpose of the book under review is to present a selection of such former standard topics in analysis, thereby making them accessible at the undergraduate level by building on the theoretical foundation provided in modern advanced calculus courses. Throughout the text, every effort has been made to include the background material necessary for full comprehension, and to maintain an elementary level as far as possible. Actually, the book evolved from a collection of notes prepared over the years for students in the author’s own advanced calculus courses, as supplements to the text and as concrete illustration of the same.

As for the precise contents, the book consists of fourteen chapters, each of which is divided into several sections. Chapter 1 is of preliminary nature and offers a rapid overview of the basics of calculus. Although proofs of the major theorems are included, it is assumed that the reader is already acquainted with most of the material. Chapter 2 deals with some special sequences in analysis, including a discussion of the number “e”, the irrationality of \( \pi \), Euler’s constant, the product formulae of Vieta and Wallis, respectively, and Stirling’s formula. Chapter 3 turns to power series and related topics, thereby touching upon topics such as Abel’s summation theorem, Taylor’s formula with remainder, Euler’s famous sum formula

\[
\sum \frac{1}{k^2} = \pi^2/6,
\]

and Weierstrass’s example of a continuous function that is not differentiable at any point. The primary goal of Chapter 4 is to develop some basic inequalities, involving sums, and integrals that are commonly applied in analysis. The reader encounters here, among several elementary inequalities, the integral analogues of the arithmetic-geometric mean inequality, Jensen’s inequality, and Hilbert’s inequality. Chapter 5 discusses the standard criteria for convergence and uniform convergence of infinite products, while Chapter 6 is devoted to the approximation of real functions by polynomials. The main focus of this chapter is the famous Weierstrass approximation theorem, for which three different proofs are given, together with Stone’s far-reaching generalization to several variables. Along the way, interpolations, Chebyshev polynomials, Bernstein polynomials, and further refinements of the Weierstrass theorem are carefully explained.

Chapter 7 explains the concept of summability for divergent series, presenting in this context some elegant and important Tauberian theorems that will be applied to other topics later in the book. Apart from A. Tauber’s original theorem [Monatsh. f. Math. 8, 273–277 (1897; JFM 28.0221.02)], the related theorems of Hardy and Littlewood, J. Karamata’s method, and Hardy’s power series are vividly depicted in this chapter.

Fourier series, the Fourier transform, and the Poisson summation formula are described in Chapter 8. The author develops criteria for convergence and Cesàro summability of Fourier series, with applications to specific (physical) examples and a discussion of the Gibbs–Wilbraham phenomenon, explains then the Fourier transform and its inversion, and concludes the chapter with the Poisson summation formula and the so-called sampling theorem. Chapter 9 develops the basic theory of the gamma function and its relation to the beta function, including Legendre’s duplication formula, Euler’s reflection formula, the...
infinite product representation, a generalization of Stirling’s formula, the Bohr-Mollerup theorem, and the calculation of a special integral related to the Riemann zeta function.

Chapter 10 discusses two unrelated topics that lie at the intersection of analysis with number theory, namely equidistributed sequences (with H. Weyl’s criterion) and the Riemann zeta function (with its relation to then gamma function and its functional equation). Chapter 11 is devoted to Bernoulli numbers, Bernoulli polynomials, general Euler sums, and the Euler-Maclaurin summation formula.

Another topic is taken up in Chapter 12, where the Cantor set is explored. The discussion in this context includes the set-theoretical notion of cardinality, Cantor’s theorem, the Schröder-Bernstein theorem, the Lebesgue measure of a set of real numbers, the construction of the Cantor set, the Cantor-Schroeder function, and the presentation of Schoenberg’s space-filling curve. Chapter 13 deals with ordinary differential equations, especially with the construction of their power series solutions at a regular singular point. After an introduction to the basic theory, including the existence and uniqueness of solutions of ordinary differential equations, power series solutions at a regular singular point of a linear differential equation of second-order are discussed in greater detail. This leads to formulas for Bessel functions and hypergeometric functions, Sturm’s method, and some refinements of Sturm’s theory of oscillation and comparison. Chapter 14 finally turns to the study of some elliptic integrals, their transformation properties, and their Legendre relation. The latter provides an effective method for numerical calculation of π. In this concluding chapter, the author also presents a classical algorithm for computing the arithmetic-geometric mean of two positive real numbers, which comes with an unexpected relation to elliptic integrals.

In general, references for each topic are given at the end of the respective chapter. Moreover, each chapter concludes with an abundance of related exercises, ranging from straightforward to more challenging, where most of them come with precise hints and programs for solution. Another feature of the book are the many historical notes and capsule scientific biographies sprinkled throughout the text, through which the origin and evolution of mathematical ideas are particularly illuminated.

All together, this book offers a wealth of gems of classical real analysis in a very lucid and exciting style of presentation. No doubt, this wonderful collection of topics is the work of a highly passionate and experienced teacher, whose expository mastery becomes apparent everywhere in the book. Originally designed for individual study, this book can also serve as a useful source for courses in advanced calculus, in particular with a view toward those topics in classical analysis that have been crowded out of the modern curriculum, on the one hand, but underlie modern developments in pure and applied mathematics nevertheless.

Summing up, this excellent book is a highly useful and valuable complement to the existing standard texts in modern advanced calculus.

Reviewer: Werner Kleinert (Berlin)

MSC:

00A05 Mathematics in general
26-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to real functions
11-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to number theory
33-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to special functions
34-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to ordinary differential equations
40-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to sequences, series, summability
41-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to approximations and expansions
42-01 Introductory exposition (textbooks, tutorial papers, etc.) pertaining to harmonic analysis on Euclidean spaces
00A35 Methodology of mathematics

Keywords:
general mathematics; real functions; classical analysis; special functions; approximation theory; Tauberian
theorems; Fourier series; Fourier transform; gamma function; Riemann zeta function; ordinary differential equations; Peano curves; elliptic integral; equidistributed sequences