The Bolzano-Weierstrass theorem is the jump of weak König’s lemma. (English)

In this paper the computational content of the Bolzano-Weierstraß theorem and variants thereof are investigated. This is done by interpreting these theorems as partial multi-valued functions and placing them in the Weihrauch lattice.

The authors start by proving basic properties of the Weihrauch lattice, and discussing closed choice and omniscience principles, compositional products, and derivatives (or: jumps) of a Weihrauch degree. Next they present their results on the Bolzano-Weierstraß theorem and the cluster-point problem. Finally, these results are compared with related results in computable analysis, constructive analysis (e.g., on the antithesis of Specker’s theorem), reverse mathematics, and proof theory.

The main results of the paper are:

1. The derivative of closed choice of a computable metric space is the cluster-point problem.
2. The Bolzano-Weierstraß theorem can be characterized as the derivative of compact choice:
   * The Bolzano-Weierstraß theorem on real numbers is the jump of weak König’s lemma.
   * The Bolzano-Weierstraß theorem on the binary space is the jump of LLPO.
   * The Bolzano-Weierstraß theorem on natural numbers is the jump of the idempotent closure of LLPO.
3. The Bolzano-Weierstraß theorem is the compositional product of weak König’s lemma and the monotone convergence theorem.
4. The Bolzano-Weierstraß theorem on real numbers is complete for the class of weakly limit-computable functions.
5. The unique-cluster-point problem on real numbers is complete for the class of functions that are limit-computable with finitely many mind changes.
6. The Bolzano-Weierstraß theorem on real numbers and, more generally, the unbounded-cluster-point theorem on real numbers are uniformly low-limit-computable.

Reviewer: Iris Loeb (Christchurch)

MSC:

03F60 Constructive and recursive analysis
03B30 Foundations of classical theories (including reverse mathematics)
03D30 Other degrees and reducibilities in computability and recursion theory
03E15 Descriptive set theory

Keywords:

computable analysis; constructive analysis; reverse mathematics; effective descriptive set theory; Bolzano-Weierstraß theorem; jump in the Weihrauch lattice; weak König’s lemma

Full Text: DOI arXiv

References:

[1] Akama, Yohji; Berardi, Stefano; Hayashi, Susumu; Kohlenbach, Ulrich, An arithmetical hierarchy of the law of excluded middle and related principles, (), 192-201


[40] Weihrauch, Klaus, Computational complexity on computable metric spaces, Mathematical logic quarterly, 49, 1, 3-21, (2003) · Zbl 1018.03049

[41] Ziegler, Martin, Revising type-2 computation and degrees of discontinuity, (.), 255-274 · Zbl 1262.03150

This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.