Summary: The three-dimensional spherical polytropic Lane-Emden problem is $y_{rr} + \frac{2}{r}y_r + y^m = 0$, $y(0) = 1$, $y_r(0) = 0$ where $m \in [0, 5]$ is a constant parameter. The domain is $r \in [0, \xi]$ where $\xi$ is the first root of $y(r)$. We recast this as a nonlinear eigenproblem, with three boundary conditions and $\xi$ as the eigenvalue allowing imposition of the extra boundary condition, by making the change of coordinate $x \equiv r/\xi$: $y_{xx} + \frac{2}{x}y_x + \xi^2 y^m = 0$, $y(0) = 1$, $y_x(0) = 0$, $y(1) = 0$. We find that a Newton-Kantorovich iteration always converges from an $m$-independent starting point $y^{(0)}(x) = \cos(\pi/2x)$, $\xi^{(0)} = 3$. We apply a Chebyshev pseudospectral method to discretize $x$. The Lane-Emden equation has branch point singularities at the endpoint $x = 1$ whenever $m$ is not an integer; we show that the Chebyshev coefficients are $a_n \sim \text{constant}/n^{2m+5}$ as $n \to \infty$. However, a Chebyshev truncation of $N = 100$ always gives at least ten decimal places of accuracy—much more accuracy when $m$ is an integer. The numerical algorithm is so simple that the complete code (in Maple) is given as a one page table.

MSC:

65L10 Numerical solution of boundary value problems involving ordinary differential equations
65L15 Numerical solution of eigenvalue problems involving ordinary differential equations
34B15 Nonlinear boundary value problems for ordinary differential equations
34L05 General spectral theory of ordinary differential operators

Keywords:
Lane-Emden problem; Chebyshev polynomial; pseudospectral method; nonlinear eigenproblem; Newton-Kantorovich iteration; branch point singularities; algorithm

Software:
Maple

Full Text: DOI