Abbassi, Mohamed T. K; Calvaruso, Giovanni; Perrone, Domenico
**Harmonic maps defined by the geodesic flow.** (English) Zbl 1251.53038
Houston J. Math. 36, No. 1, 69-90 (2010).

The Riemannian geometry of the tangent bundle $TM$ and the tangent sphere bundle $T_1M$ of a Riemannian manifold $(M,g)$ started with the works of Sasaki [*S. Sasaki*, Tôhoku Math. J., II. Ser. 10, 338-354 (1958; Zbl 0086.15003); Tôhoku Math. J., II. Ser. 14, 146–155 (1962; Zbl 0109.40505)], where the metrics used on $TM$ and $T_1M$ are the so-call Sasaki metric $g^S$ and the induced Sasaki metric ($T_1M$ is viewed as a hypersurface of $TM$). Later in the 1970's another metric called Cheeger-Gromoll metric $g^{CG}$ on $TM$ was suggested in [*J. Cheeger* and *D. Gromoll*, Ann. Math. (2) 96, 413–443 (1972; Zbl 0246.53049)] and was later constructed explicitly in [*E. Musso* and *F. Tricerri*, Ann. Mat. Pura Appl., IV. Ser. 150, 1–19 (1988; Zbl 0658.53045)]. Recently, a family of $g$-natural Riemannian metrics on $TM$ was introduced and studied (see, e.g., [*K. M. T. Abbassi* and *M. Sarih*, Differ. Geom. Appl. 22, No. 1, 19–47 (2005; Zbl 1068.53016); Arch. Math., Brno 41, No. 1, 71–92 (2005; Zbl 1114.53015)]). This 6-parameter family of Riemannian metrics includes the Sasaki metric and the Cheeger-Gromoll metric as members and can be described as

$$\begin{cases} G(x,u)(X^h,Y^h) = (\alpha_1+\alpha_3)(r^2)g_x(X,Y) + (\beta_1+\beta_3)(r^2)g_x(X,u)g_x(Y,u), \\ G(x,u)(X^h,Y^v) = \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u), \\ G(x,u)(X^v,Y^v) = \alpha_1(r^2)g_x(X,Y) + \beta_1(r^2)g_x(X,u)g_x(Y,u), \end{cases}$$

where $r^2 = g_x(u,u)$.

The harmonicity of the map $V:(M,g)\longrightarrow(T_1M,g^S)$ defined by a unit vector field on $M$ has been studied by several authors including *G. Wiegmink* [Math. Ann. 303, No. 2, 325–344 (1995; Zbl 0834.53034)], *C. M. Wood* [Geom. Dedicata 64, No. 3, 319–330 (1997; Zbl 0878.58017)], and *D. -S. Han* and *J. -W. Yim* [Math. Z. 227, No. 1, 83–92 (1998; Zbl 0891.53024)]. If the base manifold $(M,g)$ is two-point homogeneous, then it was proved by *E. Boeckx* and *L. Vanhecke* [Differ. Geom. Appl. 13, No. 1, 77–93 (2000; Zbl 0973.53053)] that the map $\bar{\xi}:(T_1M,g^S)\longrightarrow(T_1T_1M,(g^S)^S)$ defined by any geodesic flow vector field is harmonic.

Replacing the Sasaki metric $g^S$ by a $g$-natural Riemannian metric $\tilde{G}$ on the tangent sphere bundle the authors of the paper under review prove that the map $\bar{\xi}:(T_1M,\tilde{G})\longrightarrow(T_\rho T_1M,\tilde{\tilde{G}})$ is always a harmonic vector field, i.e., a critical point of the energy restricted to the set of smooth unit vector fields. They also find a necessary and sufficient condition for such a map to be harmonic. Some explicit examples of such maps are also given.

Reviewer: Ye-Lin Ou (Commerce)

**MSC:**

53C43 Differential geometric aspects of harmonic maps
53D25 Geodesic flows in symplectic geometry and contact geometry
58E20 Harmonic maps, etc.



**Full Text:** Link