

Hilbert, David

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David Hilbert's lectures on the foundations of arithmetic and logic, 1917–1933. In collaboration with Ulrich Majer and Dirk Schlimm. (English) [Zbl 1275.03002](#)

David Hilbert's Lectures on the Foundations of Mathematics and Physics 1891–1933 3. Berlin: Springer (ISBN 978-3-540-20578-4/hbk; 978-3-540-69444-1/ebook). xxv, 1062 p. (2013).

Here we encounter mathematical logic and proof theory *in statu nascendi* in the lectures Hilbert held between 1917 and 1933, containing insights into the history of the subject that cannot be read from any published source during the first half of the 20th century. Far from having abandoned in 1904 the axiomatic point of view and the metamathematical investigations started in the work leading to the *Grundlagen der Geometrie* of 1899, only to return to it in 1921, as the published record and the record of Hilbert's talks would indicate, following the trail of the lectures held in Göttingen reveals that Hilbert never stopped thinking about metamathematical matters after 1904, and held lectures on foundational matters regularly. This third volume of Hilbert's foundational lectures sets off in 1917, the date that marks a break, the beginning of a new phase in Hilbert's thinking, that can be said to begin with his lecture on September 11, 1917 to the Swiss Mathematical Society in Zürich, published as the famous *Axiomatisches Denken* in 1918. It was then and there that he invited Paul Bernays to return to Göttingen as his assistant. Bernays's presence in Göttingen is an essential part of the story of mathematical logic and proof theory.

The lectures are grouped into four chapters: (1) On the principles of mathematics (from 1917/18), together with Bernays's unpublished *Habilitationsschrift* of 1918, *Beiträge zur axiomatischen Behandlung des Logik-Kalküls*; (2) On logic (from 1920); (3) On proof theory (1921/22–1923/24); (4) On the infinite (1924/25, 1931, 1933). Appendices contain reprints of the texts most relevant to those of the lectures. They are: [*D. Hilbert* and *W. Ackermann*, *Grundzüge der theoretischen Logik*. Berlin: J. Springer (1928; [JFM 54.0055.01](#))], as well as the text of several lectures held by *D. Hilbert* (in one case together with a discussion by Weyl and Bernays, as well as a letter from the latter to the former), and published in [*Abh. Math. Semin. Univ. Hamb.* 6, 65–85 (1928; [JFM 54.0055.02](#)); *Math. Ann.* 104, 485–494 (1931; [JFM 57.0054.04](#)); *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* 1931, 120–125 (1931; [JFM 57.0055.01](#)); *Math. Ann.* 102, 1–9 (1929; [JFM 55.0031.01](#))]. There is a complete list of the titles of Hilbert's lecture courses between 1886 and 1934.

The 1920 lectures on the principles of mathematics are characterized by a very detailed motivation for the introduction of any new notion, for the extension of any logical calculus to a higher level, of the kind one would find today only in a comprehensive textbook of philosophical logic. The metamathematical concerns are, much like in the *GdG*, consistency and (in)completeness. The third concern of the *GdG*, the independence of axioms, is the central concern of Bernays's *Habilitationsschrift* in the case of propositional logic. It also contains, proved without any fanfare, what one would see today as a more important result, the completeness of the propositional calculus, in both the semantical sense and as Post-completeness (*avant Post*, a matter that was first signaled by *R. Zach* in [*Bull. Symb. Log.* 5, No. 3, 331–366 (1999; [Zbl 0942.03003](#)))].

The lectures in (2) and (3) constitute a graduate introduction to proof theory, leading to first attempts at proving the consistency of weak systems of arithmetic, with the aim of tackling the consistency of arithmetic with induction by similar methods. The weak arithmetic is based on addition, the symbol 1 , the $=$ symbol (without any significance assumed, i.e. as a non-specified binary relation), two rules of inference (modus ponens and substitution for numerals (variable-free terms)), and the axioms $1 = 1$, $a = b \rightarrow a + 1 = b + 1$, $a + 1 = b + 1 \rightarrow a = b$, $a = b \rightarrow (a = c \rightarrow b = c)$, $a + 1 \neq 1$. From this, Hilbert shows why $a + 1 = 1$ cannot be derived, and presents a similar proof that a weak form of induction is consistent for an axiom system in a language extended by a unary predicate Z (meant to designate that a variable is a numeral), to which the axioms $Z(1)$ and $Z(a) \rightarrow Z(a + 1)$ have been added (and in subsequent lectures there are additional Z -axioms). The weak form of induction is here an inference schema, allowing one to deduce $Z(a) \rightarrow F(a)$ from $F(1)$ and $F(a) \rightarrow F(a + 1)$. They also contain an in-depth critique of the logicist programme, and the indispensability of the axiomatic method, which

finds an eloquent characterization in “Das Wesen dieser Methode besteht darin, dass man sich über die Voraussetzungen und die Methoden des Schliessens klar wird, die man in einer Wissenschaft gebraucht. Axiomatisch zu verfahren ist also nichts anderes, als mit Bewusstsein zu denken.” (p. 363)

The lectures in (4) are addressed to a general audience (and were intended to be *allgemeinverständlich*), and are devoted to showing the central position of the infinite in mathematics, as well as in sciences depending on mathematics. They are very much worth reading today, as they explain with great clarity, in beautiful prose, using a wide variety of examples from physics and biology, the necessity of the road traveled in the history of mathematics, and in particular the necessity of going beyond the finite. Here Hilbert points out that the world around us shows no sign of infiniteness, neither in the sense of a homogeneous and indefinitely divisible continuum, nor in the sense of the infinitely large (although there cosmology had not yet provided a clear answer, only one of likelihood), but that we are nevertheless forced to make this un-*natural* assumption in the process of modeling nature. In his own words: “ein homogenes Kontinuum, das die fortgesetzte Teilbarkeit zuliesse und somit das Unendlich-Kleine realisieren würde, [wird] in der Wirklichkeit nirgends angetroffen. Die unendliche Teilbarkeit eines Kontinuums ist nur eine in Gedanken vorhandene Operation, nur eine Idee, die durch unsere Beobachtungen der Natur und die Erfahrungen der Physik und Chemie widerlegt wird.” (p. 705) It is here that we also find the lecture “Ueber die Grundlagen des Denkens” of c. 1931, in which he introduces his version of an ω -rule: “The statement $(\forall x)A(x)$ is correct if $A(z)$ is correct whenever z is a numeral.” (p. 766)

The totality of the lectures also offers a solid basis for judging aspects of the intuitionism-formalism debate, and to determine whether and how much of Brouwer’s contention that formalism learned from intuitionism is accurate. What transpires from these lectures is that Hilbert clearly considered formalism to be only a tool devised to prove consistency, and not a philosophy of mathematics, and that he was guided in all his choices by actual mathematical practice. One can find passages that almost sound as if he were an intuitionist, such as in (c. 1931): “Erfahrung und reines Denken sind die Quellen unserer Erkenntnis. Um diese auszuschöpfen, bedienen wir uns einer gewissen Fähigkeit unseres Geistes, durch die wir schon im voraus a priori in der Vorstellung konkrete Objekte unmittelbar erleben, sodass dieselben für uns vollkommen in allen Teilen überblickbar sind und ihre Aufweisung, ihre Unterscheidung, ihre Aufeinanderfolge oder ihr Nebeneinandergereihtsein im Endlichen anschaulich da ist als etwas, das sich weder auf etwas anderes reduzieren lässt noch einer solchen Reduktion bedarf.” (p. 765)

Reviewer: [Victor V. Pambuccian \(Phoenix\)](#)

MSC:

- [03-03](#) History of mathematical logic and foundations
- [01A60](#) History of mathematics in the 20th century
- [01A75](#) Collected or selected works; reprintings or translations of classics
- [03B30](#) Foundations of classical theories (including reverse mathematics)
- [03F03](#) Proof theory in general (including proof-theoretic semantics)
- [03F30](#) First-order arithmetic and fragments

Cited in **8** Reviews
Cited in **13** Documents

Biographic references:

[Hilbert, David](#)

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