Hoffmann, T.; Rossman, W.; Sasaki, T.; Yoshida, M.
Discrete flat surfaces and linear Weingarten surfaces in hyperbolic 3-space. (English)

Away from umbilic points of a smooth minimal (or CMC) immersed surface \( f(x,y) \) into \( \mathbb{R}^3 \) or \( \mathbb{H}^3 \) one can take isothermic coordinates \((x,y)\), that are isothermal and diagonalize the second fundamental form for the induced metric. In the Euclidean case, the stereographic projection of the Gauss map defines a holomorphic function \( g \) on the complex plane that, with the Hopf differential, constitutes data to define the Weierstrass representation of \( f \). The derivative of \( g \) with \( x \) and \( y \) defines its Weierstrass data. This requires to define a normal at vertices of \( f \) with circulant quadrilaterals. A discrete caustic surface of a flat discrete one is also defined by following the perquadric of the Minkowski 4-space \( f \).

Particularly, the authors give an alternative way to define them by \( \alpha \) and \( \gamma \). For the induced metric, the stereographic projection of the Gauss map defines a new lift \( F \) of a smooth isothermically-parametrized CMC 1 surface \( f_1 : M \to \mathbb{H}^3 \) and Bryant’s equation on \( dF \), expressed in terms of an holomorphic function \( g \) (Weierstrass data) with non-zero derivative, one obtains \( f_1 = F \cdot \bar{F}^T \) as shown by R. L. Bryant in [Astérisque 154–155, 321–347 (1988; Zbl 0635.53047)].

To each \( f_1 \) there is a related flat surface \( f_0 \) with singularities, and a suitable change of coordinates leads to an equation satisfied by the components of the lift that in some particular cases is the Airy equation. One-parameter families of deformations \( f_t \) through linear Weingarten surfaces of Bryant-type between \( f_0 \) and \( f_1 \) are considered and non-uniqueness is shown for this type of deformations. A discretization of such surfaces can be built using the light cone model space of \( \mathbb{H}^3 \) in \( \mathbb{R}^3,1 \) and quaternionic notation, as shown by U. Hertrich-Jeromin in [Manusc. Math. 102, No. 4, 465–486 (2000; Zbl 0979.53008)].

In Theorem 4.2, the authors give an alternative way to define them by \( f_1 = (1/\det F)F \bar{F}^T \) (up to rigid motion in \( \mathbb{H}^3 \)) following Bryant’s formula \( f_t - f_0 = f_t \cdot G(g_\nu, q_\nu)\lambda \alpha \), where \( G(g_\nu, q_\nu) \) is a matrix depending only on \( g_\nu \) and \( q_\nu \) with a discrete holomorphic function \( g \). \( \alpha \) is the cross ratio factorization function and \( \lambda \) a free real parameter. Multiplication of \( F \) by a suitable matrix depending on \( g \) defines a new lift \( E \) that represents \( f_0 \) discrete and flat in \( \mathbb{H}^3 \), and it is proved in Theorem 4.6 that \( f_0 \) has concircular quadrilaterals. The authors also describe a deformation family \( f_t \) of discrete linear Weingarten surfaces from \( f_0 \) to \( f_1 \) with concircular quadrilaterals. A discrete caustic surface of a flat discrete one is also defined by following the smooth in terms of the Weierstrass data. This requires to define a normal at vertices of a discrete flat surface and it is shown that negativity of \( \lambda \alpha \) is equivalent to the normal geodesics in \( \mathbb{H}^3 \) emanating from two adjacent vertices \( f_0 \) and \( f_0 \) that intersect (in a unique point), or equivalently the edge \( pq \) is vertical. The set \( C_f \) of such intersection points is a discrete surface named the caustic or focal surface of \( f \), not necessarily flat. A lift \( E(C_f) \) is defined for each vertical edge \( pq \) satisfying the formula \( C_f = (1/\det(E(C_f)))E(C_f) \cdot \bar{E(C_f)}^T \). Discrete caustics have properties similar to that of caustics in the smooth case. Many examples are described throughout the paper, with some beautiful figures, and special attention is given to the discrete flat surface for the case \( g = z^{3/4} \), related to the Airy equation.

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References:


[20] Masatoshi Kokubu, Wayne Rossman, Masaaki Umehara, and Kotaro Yamada, Asymptotic behavior of flat surfaces in hyper-
References:


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