Chan, Melody

Combinatorics of the tropical Torelli map. (English) Zbl 1283.14028

Algebra Number Theory 6, No. 6, 1133-1169 (2012).

This paper serves essentially a three-fold purpose:

(1) it provides a very readable and gentle introduction to the key objects and definitions arising in the moduli spaces of tropical curves and abelian varieties, largely a summary of Mikhalkin et al.'s work [*G. Mikhalkin*, Not. Am. Math. Soc. 54, No. 4, 511–513 (2007; Zbl 1142.14300); *G. Mikhalkin* and *I. Zharkov*, Contemp. Math. 465, 203–230 (2008; Zbl 1152.14028)] and the subsequent development and refinements by Caporaso, Viviani et al. [*S. Brannetti* et al., Adv. Math. 226, No. 3, 2546–2586 (2011; Zbl 1218.14056); *L. Caporaso* and *F. Viviani*, Duke Math. J. 153, No. 1, 129–171 (2010; Zbl 1200.14025)], etc.;

(2) it puts forth a minor modification to a definition appearing in these latter works, namely that of a "stacky fan," in order to fix a gap and render some basic claims in those papers correct;

(3) it uses computational methods, elementary combinatorics, and some deeper pre-existing combinatorial results (some intriguingly classical ones, in particular, pre-dating tropical geometry!) to prove some basic results on these tropical moduli spaces and provide explicit descriptions of their combinatorial structure in low genus.

Let us now elaborate slightly on these three aspects. The perspective taken on these tropical objects (curves and abelian varieties) and their moduli spaces is a now widely-accepted one that is both intrinsic yet ad hoc. In other words, tropicalization per se does not play a role, rather it is used simply to motive the definition of tropical curves simply as decorated graphs, and similarly for tropical abelian varieties. In this way, one has entirely combinatorial/polyhedral objects that assume a geometric significance through the tropical analogy with algebraic geometry. For instance, these objects have automorphisms so when parameterizing them one must introduce a type of combinatorial/polyhedral orbifold which is essentially something that is locally a balanced polyhedral complex ("tropical variety") modulo a finite group action. This leads to an ad hoc definition of a stack in the tropical word (quite primitive in many ways, as descent and gluing play no role, one simply wants to remember automorphisms pointwise, families are not considered) that nonetheless seems adequate for many purposes. This ad hoc notion was introduced and termed a "stacky fan" (related to, though distinct from, the situation of toric varieties where a stacky structure can be encoded in the defining fan) by Caporaso and Viviani et al. in the above-cited papers, though in this paper Chan astutely points out a deficiency in their definition: it disallows the moduli spaces of tropical curves and tropical abelian varieties, what are supposed to be the principal examples of tropical stacks! Chan's modification is minor and still ad hoc, but suffices to correct this inadequacy. One hopes a more rigorous notion of stack/moduli space will eventually be introduced to the tropical literature, but for now the present definition still permits a nice combinatorial study of these objects. Along the way to proving that the standard constructions of the moduli spaces of tropical curves and abelian varieties are indeed stacky fans in this new sense, Chan provides some nice elementary examples of the objects/constructions.

In a more extensive example, Chan computes the poset structure of the tropical moduli spaces of curves for genus $g \leq 5$. This is accomplished by a simple algorithm described in the paper, implemented in Mathematica, and with accompanying numerical data posted on the author's website. The paper also includes a combinatorial study of a locus in the moduli space of tropical abelian varieties containing the image of the tropical Torelli map (i.e., this locus includes in particular all tropical Jacobians of graphs). Computational methods return to describe the poset structure of the tropical Schottky locus, namely, the locus of tropical Jacobians. This is possible due to a convenient fact that these have a matroidal interpretation which ties them into pre-existing literature, and also due to a clever algorithmic approach described by Chan in this paper. Finally, the paper concludes with some preliminary investigation into rigidifying these tropical stacks by considering level structures.

Reviewer: Noah Giansiracusa (Zürich)

© 2021 FIZ Karlsruhe GmbH

MSC:

14T05	Tropical geometry (MSC2010)
14H10	Families, moduli of curves (algebraic)
05C30	Enumeration in graph theory

Cited in **14** Documents

Keywords:

tropical geometry; tropical curves; metric graphs; Torelli map; moduli of curves; abelian varieties

Software:

Mathematica

Full Text: DOI Link arXiv

© 2021 FIZ Karlsruhe GmbH