Borwein, Jonathan M.; Straub, Armin; Wan, James; Zudilin, Wadim
Densities of short uniform random walks. (English) Zbl 1296.33011
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An n-step random walk on the plane is a random walk starting at the origin and consisting an n consecutive steps of length 1 each taken into a uniformly random direction.

Let $p_n(x)$ denote the radial density of the distance travelled in n steps. There is no known closed form expression for p_n in general, however it is elementary to show that

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}} \quad (0 \le x \le 2).$$

The expression for p_3, due to Pearson, is much more complicated:

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2} K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right) \quad (0 \le x \le 3).$$

The present authors find a hypergeometric expression and detailed singularity analysis for p_3 and p_4. One of the main theorems of the paper is the following new hypergeometric representation:

$$p_4(x) = \frac{2\sqrt{16-x^2}}{\pi^2 x} \operatorname{Re}\left({}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ \frac{5}{6}, \frac{7}{6}\end{matrix}\right| \frac{(16-x^2)^3}{108x^4}\right)\right) \quad (0 < x < 4).$$

Some general considerations for all the p_n's are also made: it is shown that they satisfy certain differential equations. Relations to Mahler measures are also discussed.

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MSC:

33C20	Generalized hypergeometric series, ${}_pF_q$
60G50	Sums of independent random variables; random walks
34M25	Formal solutions and transform techniques for ordinary differential equations in the complex domain
44A10	Laplace transform
05A19	Combinatorial identities, bijective combinatorics
11F11	Holomorphic modular forms of integral weight

Keywords:
random walks; hypergeometric functions; Mahler measure; Mellin transform of moments

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