Gill, James T.

Doubling metric spaces are characterized by a lemma of Benjamini and Schramm. (English)


The author gives a new characterization of doubling metric spaces in terms of an abstract version of a geometric property of Euclidean space first identified by Benjamini and Schramm.

Let \((X, d)\) be a metric space and \(S \subset X\) a finite subset. The isolation radius \(\rho_w\) of a point \(w \in S\) is the minimum distance from \(w\) to any other point of \(S\).

I. Benjamini and O. Schramm [Electron. J. Probab. 6, Paper No. 23, 13 p. (2001; Zbl 1010.82021)] prove the following geometric fact: for every \(0 < \delta < 1\) there exists \(C = C(\delta) > 0\) so that for each finite set \(S \subset \mathbb{R}^2\) and for each \(t \geq 2\),

\[\left| \{ w \in S : \inf_{p \in X} \left| \{ x \in S : d(x, w) \leq \rho_w / \delta \text{ and } d(x, p) \geq \delta \rho_w \} \geq t \right| \right| \leq C \frac{\# S}{t}. \tag{1}\]

Here \(\# A\) denotes the cardinality of a finite set \(A\). The analogous result holds also in Euclidean space \(\mathbb{R}^n\) of any dimension, in which case the constant \(C\) depends on both \(\delta\) and \(n\).

The author makes the following definition: A metric space \((X, d)\) carries a Benjamini-Schramm lemma if for every \(0 < \delta < 1\) there exists \(C = C(X, \delta) > 0\) so that for each finite set \(S \subset X\) and for each \(t \geq 2\), the estimate in (1) holds true. Recall that a metric space \((X, d)\) is said to be doubling if there exists a constant \(M\) so that every ball \(B(x, r)\) in \(X\) can be covered by at most \(M\) balls of radius \(r/2\).

The main result of the paper is the following: Let \((X, d)\) be a metric space. Then \((X, d)\) carries a Benjamini-Schramm lemma if and only if \((X, d)\) is doubling.

To show that doubling metric spaces carry a Benjamini–Schramm lemma, the author shows that the validity of such a lemma is invariant under snowflake transformations and under bi-Lipchitz transformations, and then appeals to Assouad’s snowflake embedding theorem. The converse direction is more difficult. For a nondoubling space \((X, d)\), the author provides an explicit construction of a family of subsets \(S_N \subset X\) such that \(\# S_N > N\) and for all \(w \in S_N\) and all \(p \in X\),

\[\left| \{ x \in S : d(x, w) \leq 40 \rho_w \text{ and } d(x, p) \geq \frac{1}{40} \rho_w \} \right| \geq N.\]

The existence of such a family of subsets contradicts the assumed validity of a Benjamini-Schramm lemma in \((X, d)\), completing the proof.

Reviewer: Jeremy Tyson (Urbana)

MSC:

30L05 Geometric embeddings of metric spaces
28A75 Length, area, volume, other geometric measure theory

Keywords:

doubling space; Assouad dimension

Full Text: DOI arXiv

References:
