A classical theorem of differential geometry states that every connected surface all of whose points are umbilic is a piece of a round sphere or a plane. Usually, in elementary courses of differential geometry this theorem is proved for Euclidean 3-space but the same theorem can be proved in n-space for all n ≥ 3 by the same method. Some authors attribute this theorem to Gaston Darboux (1842–1917) (see, e.g., page 336 in [Yu. G. Reshetnyak, Stability theorems in geometry and analysis. Dordrecht: Kluwer Academic Publishers (1994; Zbl 0925.53005)], while others do not link it with the name of any geometry (see, e.g., page 217 in [S. S. Chern, W. H. Chen and K. S. Lam, Lectures on differential geometry. Singapore: World Scientific (1999; Zbl 0940.53001)]. The authors of the paper under review attribute this classical theorem to Delfino Codazzi (1824–1873), but do not provide any reference, nor any argument to support their point of view.

According to Reshetnyak [loc. cit.], it was M. A. Lavrent’ev (1900–1980) who first raised the problem of stability in the Darboux theorem. Omitting technical details, we can formulate this problem as follows: Is it true that every nearly umbilic surface is nearly a sphere? Pioneer results on this stability problem were obtained for convex nearly umbilic hypersurfaces in Euclidean n-space. Among those pioneer results on convex nearly umbilic hypersurfaces, the most general one was obtained by S. K. Vodop’yanov in [Sib. Math. J. 11, No. 5, 724–735 (1970; Zbl 0202.21502)]. Using the method of integral representations, he obtained quantitative estimates of the deviation of nearly umbilical hypersurfaces from a sphere in $W_2^2$-norms, $p > 1$. Vodop’yanov’s theorem is presented in detail in Chapter 6 of the above mentioned book by Yu. G. Reshetnyak.

The authors of the paper under review completely omit the history of the problem they are dealing with. They start the exposition with theorems of C. de Lellis and S. Müller published in [J. Differ. Geom. 69, No. 1, 75–110 (2005; Zbl 1087.53004)]. Those theorems read that, for every smooth embedded surface $\Sigma \subset \mathbb{R}^3$ with area normalized to $H^2(\Sigma) = 4\pi$, the second fundamental form $A_\Sigma$ of $\Sigma$ satisfies $\|A_\Sigma - \text{id}\|_{L^2(\Sigma)} \leq C\|A_\Sigma\|_{L^2(\Sigma)}$ and that $\Sigma$ is close to a round sphere in $W_2^2$, when $\|A_\Sigma\|_{L^1(\Sigma)}$ is small. Here $A_\Sigma$ denotes the tracefree part of the second fundamental form of $\Sigma$, and $\text{id} : S^2 \subset \mathbb{R}^3 \to \mathbb{R}^3$ is the standard isometric embedding of the round sphere.

In the paper under review, the authors use a new method based on the monotonicity formula for varifolds and extend the above results of C. de Lellis and S. Müller to arbitrary codimension. The main results read as follows:

**Theorem 1.1:** Let $\Sigma \subset \mathbb{R}^n$, $n \geq 3$, be a two-dimensional smoothly embedded closed connected surface with $H^2(\Sigma) = 4\pi$. Then there exists a measurable unit normal vector field $N$ on $\Sigma$ with

$$\|A_\Sigma - N\delta g\|_{L^2(\Sigma)} \leq C_n\|A_0\|_{L^2(\Sigma)}$$

and

$$\|K_\Sigma - 1\|_{L^1(\Sigma)} \leq C_n\|A_0\|_{L^2(\Sigma)}$$

where $K_\Sigma$ denotes the Gauss curvature of $\Sigma$.

**Theorem 1.2:** Let $\Sigma \subset \mathbb{R}^n$, $n \geq 3$, be a two-dimensionally smoothly embedded surface of sphere type with $H^2(\Sigma) = 4\pi$ and $\|A_\Sigma\|_{L^2(\Sigma)}^2 < 2e(n)$, where $e(3) = 8\pi$, $e(4) = 16\pi/3$, and $e(n) = 4\pi$ for $n \geq 5$. Then there exists a conformal parametrization $f : S^2 \to \mathbb{R}^n$ with pull-back metric $g = f^*g_{\text{euc}} = e^{2\tau}g_{S^2}$, such that after an appropriate translation and rotation and with $S^2 := \partial B_1(0) \cap \mathbb{R}^3 \subset \mathbb{R}^n$

$$\|f - \text{id}_{S^2}\|_{W_2^2(S^2)} + \|u\|_{L^\infty(S^2)} \leq C(n, \tau)\|A_\Sigma\|_{L^2(\Sigma)}$$

where $\tau := 2e(n) - \|A_\Sigma\|_{L^2(\Sigma)}^2 > 0$.


Reviewer: Victor Alexandrov (Novosibirsk)
References:


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