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Evaluation of the dimension of the $\mathbb{Q}$-vector space spanned by the special values of the Lerch function. (English) Zbl 1323.11052

Tsukuba J. Math. 38, No. 2, 171-188 (2014).

For $x \in \mathbb{R}$ with $x > 0$, $k \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|z| \leq 1$ and $(k, z) \neq (1, 1)$, the Lerch zeta function is defined by

$$L(x, z;\ k) := \sum_{m=0}^{\infty} \frac{z^{m+1}}{(m+x)^k}.$$

When $x = 1$, $L(x, z;\ k) = Li_k(z)$, the polylogarithm function. A classical result of *E. M. Nikishin* [Math. USSR, Sb. 37, 381–388 (1980; Zbl 0441.10031)] established linear independence of certain special values polylogarithm functions over the field of rational numbers.

In this article, the author generalizes the above result for certain special values of the Lerch zeta functions. In particular, the statement of main theorem is as follows; Let $k \in \mathbb{N}$, $x = \frac{\alpha}{\beta} > 0$ and $z = \frac{a}{b} < -1$ where $\alpha, \beta, a, b \in \mathbb{N}$ with $\beta, b > 0$. If the inequality

$$b^{k+1} \ < \ |a| \exp\Bigg(- \Bigg[k^2 \Bigg(\log \beta + \sum_{\substack{p|\beta \\ p \text{ prime}}} \frac{\log p}{p-1} + \beta \Bigg) + \ (k-1)(k \log k + (2k+1) \log 2) - k \Bigg] \Bigg)$$

holds, then the numbers

$$1,\ L\left(x,\ \frac{1}{z};\ j\right), \quad 1 \leq j \leq k$$

are linearly independent over the field of rational numbers.

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MSC:

11J72 Irrationality; linear independence over a field
11M35 Hurwitz and Lerch zeta functions
11J81 Transcendence (general theory)

Keywords:

Padé approximation; linear independence over rational numbers, Lerch zeta function

Full Text: DOI Euclid