One of the primary motivations, and stunning achievements, of Mumford’s development of Geometric Invariant Theory (GIT) was to construct the moduli space of curves. The process is to pluri-canonically embed each curve of a fixed genus in projective and then parameterize these embedded curves as a locally closed subset of a Hilbert scheme; since two such curves are isomorphic if and only if they differ by an automorphism of this ambient projective space, the moduli space of curves is then the quotient of this locally closed locus by $\text{PGL}$. GIT is used to determine a semistable locus for this action, and there are essentially three choices involved: there is the choice of what $m$ to use when $m$-canonically embedding the curve, there is the choice between using a Hilbert scheme and a Chow variety to parameterize the embedded curves, and for the Hilbert scheme there is the choice of a polarization. In Mumford’s days (e.g., in the work of Mumford and Gieseker) the challenge was to choose these parameters so that the resulting GIT semistable curves are precisely the now ubiquitous moduli of stable (i.e., nodal with finite automorphism group) curves. However, as progress in birational geometry gained momentum and variational GIT (VGIT) developed, this abundance of choices/parameters was recognized as a blessing in disguise: different choices lead to different moduli spaces which are birational to the standard one and whose modular interpretations are closely related. The Hassett-Keel program took this perspective into a powerful direction: run the log minimal model program on the Deligne-Mumford moduli space of stable curves and recognize the steps in the program as modifications of the moduli functor, usually by allowing worsening singularities as one proceeds. This is a fertile area with beautiful progress by many researchers.

The present research monograph studies a closely related but distinct problem: rather than looking at GIT stability for $\text{PGL}$ acting on the locally closed subset of the Hilbert scheme parameterizing pluri-canonical embedded curves, the authors look at the entire Hilbert scheme (or the main irreducible component) of curves of a fixed degree $d$ and genus $g$ in a projective space of dimension $d - g$. The authors are interested in, roughly, the ratio $d$ to $g$ (technically, it is not quite this but a ratio of this flavor). For large values stability was worked out by Lucia Caporaso in her PhD thesis, so the present question is what happens as the ratio decreases. There are critical values and walls crossed at these, and the present book provides a detailed analysis, identifying the walls, determining both Hilbert and Chow stability, and describing the resulting quotients from a moduli perspective in terms of what objects are allowed and disallowed at each juncture. Thus, while not literally an instance of the LMMP as in the Hassett-Keel program, this program has a quite similar flavor and shares remarkably many properties and examples. It is an important contribution to the literature. The main techniques are an extension of those of Mumford, Gieseker, and Caporaso, though pulling off this technical feat naturally requires a lot of work and creativity. One application of this particular GIT problem is a corresponding collection of compactifications of the universal Jacobian; this was the main motivation for Caporaso’s original work and it is featured toward the end of this book.

Reviewer: Noah Giansiracusa (Athens)

MSC:

- 14L24 Geometric invariant theory
- 14L30 Group actions on varieties or schemes (quotients)
- 14H45 Special algebraic curves and curves of low genus
- 14-02 Research exposition (monographs, survey articles) pertaining to algebraic geometry

Keywords:

- GIT; moduli; curves; stability; Hilbert; Chow

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