For a topological group $G$ and a ring $R$ with unity, let $R[G]$ denote the $R$-group algebra. $G$ is said to be of type OS if the set of open subgroups of $G$ is a basis of neighborhoods of 1 in $G$. For example, any profinite group is of type OS. For such a $G$, the authors introduce the category of discrete $R[G]$-modules, which is an abelian category with enough injectives, which allows one to define and study discrete cohomology (or more generally, extensions). The category admits restriction, coinduction, and inflation functors, and the authors discuss basic properties of this discrete cohomology theory. For a discrete group $G$, the discrete cohomology of $\mathbb{Z}[G]$ agrees with the ordinary group cohomology of $G$.

Per the title, much of the focus of the paper is on the case when $G$ is a totally disconnected, locally compact (t.d.l.c.) group and the ring $R$ is assumed to be the rational field $\mathbb{Q}$. Such a group is known to be of type OS. If $G$ is in fact compact (equivalently profinite), then all discrete modules are both injective and projective, giving trivial cohomology. In general, for an arbitrary t.d.l.c. group $G$, the category of discrete $\mathbb{Q}[G]$-modules has enough projectives, allowing the authors to discuss the notion of projective dimension and related concepts. In this context, rational discrete homology is also defined.

The authors construct a standard discrete $\mathbb{Q}[G]$-bimodule $\text{Bi}(G)$ which is defined as a direct limit of group algebras $\mathbb{Q}[G/O]$ over open, compact subgroups $O$ of $G$. For a finitely generated projective module $P$ and arbitrary module $M$, one has a key identity: $\text{Hom}_G(P, \text{Bi}(G)) \otimes_G M \simeq \text{Hom}_G(P, M)$. Using this, the authors introduce the notion of a t.d.l.c. group being a rational duality group of dimension $d$: being of type $FP$ and having discrete cohomology $\text{dH}^k(G, \text{Bi}(G)) = 0$ for all $k \neq d$. The module $\text{dH}^d(G, \text{Bi}(G))$ is referred to as the rational dualizing module. For such groups, one has a duality relationship between cohomology and homology.

The authors further study actions of t.d.l.c. groups on graphs and simplicial complexes, ultimately obtaining a criterion involving a simplicial $G$-complex for $G$ to be a rational duality group. This criterion is then applied to the group $G(K)$ of $K$-rational points of a semi-simple, simply connected algebraic group $G$ over a non-discrete non-archimedean local field $K$. It is shown that $G(K)$ is a rational t.d.l.c. group of dimension equaling the rank of $G$, and the dual of the rational dualizing module can be identified using cohomology of the topological realization of the Tits realization of the affine building associated to $G$. The authors also investigate topological Kac-Moody groups, obtaining a criterion for rational duality involving that of the associated Weyl group. This is applied to show that certain topological Kac-Moody groups are indeed rational duality groups, but examples are also given of topological Kac-Moody groups that are not rational duality groups.

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