Merkurjev, A.

Weight two motivic cohomology of classifying spaces for semisimple groups. (English)


This paper is concerned with computations in the étale motivic cohomology of $\mathbb{Z}(2)$ for semisimple groups, their classifying spaces and torsors.

Let $F$ be any field and $F_{\text{sep}}$ be a separable closure. For an $F$-variety $X$ and integers $i \geq 0$ and $n \geq 0$, one can define integral étale motivic cohomology groups, which following Merkurjev we make semisimple (which we make in the rest of this review). In the case where $X$ is a semisimple algebraic group over $F$ and $f : X \to Y$ be a $G$-torsor. By a result of J. J. Sansuc [J. Reine Angew. Math. 327, 12–80 (1981; Zbl 0468.14007)], there is an exact sequence

$$0 \to H^1(Y,\mathbb{Z}(1)) \to H^1(X,\mathbb{Z}(1)) \to \hat{G} \to H^2(Y,\mathbb{Z}(1)) \to$$

$$H^2(X,\mathbb{Z}(1)) \to H^2(G,\mathbb{Z}(1)) \to H^3(Y,\mathbb{Z}(1)) \to H^3(\hat{G},\mathbb{Z}(1))$$

This exact sequence, applied to the algebraic approximation of the universal torsor over $BG$, implies classical results on the étale motivic cohomology of $\mathbb{Z}(1)$ for $BG$ in low degrees, namely

$$H^i(BG,\mathbb{Z}(1)) \simeq \begin{cases} K_1(F) = F^\times & i = 1 \\ \hat{G} & i = 2 \\ \text{an extension of a subgroup of } Br(F) \text{ by } Pic(G) & i = 3 \end{cases}$$

This paper is concerned with analogues of these results for $\mathbb{Z}(2)$, under the extra assumption that $G$ is semisimple (which we make in the rest of this review). In the case where $G$ is simply connected, the results of the paper take a very simple form, so it makes sense to start with this case. Under this extra assumption, the first main theorem 5.1 states that the natural map

$$H^2(Y,\mathbb{Z}(2)) \to H^2(X,\mathbb{Z}(2))$$

is an isomorphism, and that in the next two degrees there is an exact sequence

$$0 \to H^3(Y,\mathbb{Z}(2)) \to H^3(X,\mathbb{Z}(2)) \to Q(G) \to H^4(Y,\mathbb{Z}(2)) \to H^4(X,\mathbb{Z}(2)).$$

The group $Q(G)$ is a lattice, of rank at most the number of simple factors of $G_{\text{F_{sep}}}$, which has been introduced and studied in connection with the Galois cohomological invariants of $G$ (see section 31 of [M.-A. Knus et al., The book of involutions. With a preface by J. Tits. Providence, RI: American Mathematical Society (1998; Zbl 0955.16001)].
Similarly, again assuming $G$ simply connected, the second main theorem 5.3 takes the form

$$H^i(BG, \mathbb{Z}(2)) \simeq \begin{cases} K_2(F) & i = 2 \\ 0 & i = 3 \\ \text{an extension of a subgroup of } H^4(F, \mathbb{Z}(2)) \text{ by } Q(G) & i = 4 \end{cases}$$

As explained by the author, this case ($G$ simply connected) is already written down in Appendix 4 of [H. Esnault et al., J. Am. Math. Soc. 11, No. 1, 73–118 (1998; Zbl 1025.11009)].

Let $G$ be an arbitrary semisimple group. Write $G^\text{sc}_{/\mathbb{F}}$ for a simply connected cover of $G_{/\mathbb{F}}$, and $\hat{C}$ for the Galois module $\text{Hom}(\text{Ker}(G^\text{sc}_{/\mathbb{F}} \to G_{/\mathbb{F}}), \mathbb{G}_m)$. The main results, Theorem 5.1 and 5.3, are complicated to state in full detail in this more general case, but we can summarize them by saying that, compared to the simply connected case above, they involve a few more terms which are all defined in terms of $\hat{C}$.

The results of this paper have been applied by the author in [J. Eur. Math. Soc. (JEMS) 18, No. 3, 657–680 (2016; Zbl 1367.12003)] to determine all degree 3 cohomological invariants of semisimple groups with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$.

We now say a word about the proofs. Theorem 5.3 is deduced from Theorem 5.1 applied to the algebraic approximation of the universal torsor over $BG$, together with some technical work related to the complication in the definition of $H^4(BG, \mathbb{Z}(2))$.

Let us now summarize the proof of Theorem 5.1. The most difficult part of the argument concerns what happens for cohomological degree 4. Accordingly, the beginning of the paper is devoted to a computation of $H^i(G, \mathbb{Z}(2))$ and $H^i(X, \mathbb{Z}(2))$ for $i = 2, 3$. By the formulas of Kahn, this is reformulated in terms of $K$-cohomology, and the argument is then phrased using Rost modules and the Rost spectral sequence (a relative version of the Gersten-type spectral sequence) applied to $G \to G/T$ (with $T$ maximal torus) and to $f$ itself. Then, using these results as a starting point, Merkurjev studies the complex of étale sheaves on $Y$ defined by

$$Z_f(2) := \text{Cone}(\mathbb{Z}_Y(2) \to Rf_*\mathbb{Z}_X(2))$$

and in particular constructs a distinguished triangle

$$\hat{C}(1)[1] \to \tau_{\leq 3}Z_f(2) \to D(G) \to$$

with $D(G)$ a close relative to the lattice $Q(G)$ above. This triangle together with the low degree computations above is enough to finish the proof (modulo the precise identification of the morphisms involved, part of which is done in a technical appendix).

Reviewer: Simon Pepin Lehalleur (Berlin)

**MSC:**

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**Keywords:**

motivic cohomology; semisimple groups; $K$-cohomology; Galois cohomology; torsors

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