Bifocal curves in a Euclidean plane are defined by having a constant sum, absolute difference, product or ratio of distances to two given points, the focal points. They are, in this order, ellipses, hyperbolas, Cassini curves, or circles ("Apollonius circles"). Their defining properties naturally extend to higher dimensions and, in case of constant sum or product, to finitely many focal points. In the article at hand, the authors study multifocal curves and surfaces in real vector spaces of finite dimension endowed with a non-negative, positively homogeneous, and convex function $\gamma$, the gauge, to measure length ("generalized Minkowski spaces").

Some of their results generalize known facts from Euclidean geometry. For example the bisector of two points on a (multifocal) ellipse with respect to the opposite gauge intersects the convex hull of the focal points or confocal hyperbolas and ellipses intersect (under some regularity assumptions) orthogonally in Birkhoff sense, if the unit circle is a polygon. It is also possible to characterize non-strictly convex gauges, typically by the existence of line segments on multifocal surfaces. Gauges that are inner-product norms can for example be characterised by convexity of the sublevel sets associated to hyperbolas ("interior of one branch") or by the fact that Apollonius surfaces are spheres.

Results on (multifocal) Cassini curves mostly refer to their shapes. An $n$-focal Cassini curve is star-shaped with respect to a given point if its radius is sufficiently large. For sufficiently small radius, it consist of star-shaped components around the focal points. However, the authors also present an exotic example of a Cassini curve with uncountably many components.

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MSC:
- 51M04 Elementary problems in Euclidean geometries
- 46B20 Geometry and structure of normed linear spaces
- 14H50 Plane and space curves
- 49J53 Set-valued and variational analysis
- 52A10 Convex sets in 2 dimensions (including convex curves)
- 52A21 Convexity and finite-dimensional Banach spaces (including special norms, zonoids, etc.) (aspects of convex geometry)
- 53A04 Curves in Euclidean and related spaces
- 14H45 Special algebraic curves and curves of low genus
- 53A05 Surfaces in Euclidean and related spaces

Keywords:
- Apollonius circle; bifocal curve; Birkhoff orthogonality; multifocal Cassini curve; convex distance function; multifocal ellipse; gauge; generalized hyperbola; generalized Minkowski space; multifocal curve; $n$-lemniscate; sublevel set; Voronoi cell

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