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From volume cone to metric cone in the nonsmooth setting. (English) Zbl 1356.53049

In Riemannian geometry, a volume cone is a Riemannian manifold such that the volume of metric balls $B(x, r)$ is proportional to $r^n$. On the other hand, one says that $B_R(x) \setminus B_r(x)$ is a metric cone if it is a warped product $(N \times [r, R], dt^2 + t^2 g_N)$ for some closed manifold $(N, g_N)$. A tangent cone of a Riemannian manifold $(M, g)$ means a Gromov-Hausdorff limit of $(M, r^{-2} g)$ for a sequence $r_j \to \infty$.

The volume comparison theorem for Riemannian manifolds $M$ of nonnegative Ricci curvature asserts that the volume of metric balls $B_r(x)$ grows more slowly than the volume of metric balls in Euclidean space:

$$\frac{d}{dr} \left( \frac{\text{vol}(B_r(x) \subset M)}{\text{vol}(B_r \subset \mathbb{R}^n)} \right) \leq 0.$$

The equality case $\frac{d}{dr} \left( \frac{\text{vol}(B_r(x) \subset M)}{\text{vol}(B_r \subset \mathbb{R}^n)} \right) = 0$ means exactly that one has a volume cone and the rigidity part of the volume comparison theorem asserts that if a Riemannian manifold of nonpositive Ricci curvature is a volume cone, then it is a metric cone.

In [Ann. Math. (2) 144, No. 1, 189–237 (1996; Zbl 0865.53037)], J. Cheeger and T. H. Colding proved that a Riemannian manifold with $\text{Ric} \geq 0$ and almost maximal volume must be Gromov-Hausdorff close to a metric cone. In particular, if $\text{Ric} \geq 0$ and $\text{vol}(B, x, r) \geq c r^n$ for some $c > 0$, then each tangent cone is a metric cone.

The purpose of the paper under review is to generalize these results to the non-smooth setting, that is, to metric-measure spaces which have nonpositive Ricci curvature in a synthetic sense.

A natural setting for such a generalisation would have been the $CD(0, n)$ and $CD^*(0, n)$ spaces introduced by Lott-Villani and Bacher-Sturm. However, they include some Finsler geometries for which the wanted rigidity does not always hold. For this reason the authors restrict to a class of spaces called $RCD^*(0, n)$ which has been introduced by the second author [Mem. Am. Math. Soc. 1113, iii-v, 91 p. (2015; Zbl 1325.53054)] and can be characterised by a finite-dimensional Bochner inequality.

The main result of the paper is then that for a ball in a $RCD^*(0, n)$ space, being a volume cone implies being locally isometric to the cone over a ball in an $RCD^*(n-2, n-1)$ space. (There are two exceptional cases in which the ball is just 1-dimensional.) The consequences for tangent cones will be analyzed in subsequent papers.

Reviewer: Thilo Kuessner (Seoul)

MSC:

53C23 Global geometric and topological methods (à la Gromov); differential geometric analysis on metric spaces
53C20 Global Riemannian geometry, including pinching

Keywords:

bounded Ricci curvature; rigidity theorems; warped product; metric geometry; optimal transport; RCD(0,N) condition; curvature-dimension conditions; volume comparison

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References:


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