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Characterization of Radon integrals as linear functionals. (English. Russian original)
Zbl 1358.28010

Many mathematicians contributed to the theory of integration creating a variety of different techniques and approaches in this area and building integrals of different forms, types and structures. It is possible to classify this diversity of integrals inducing a definite order in their multiplicity. The main two classes of integrals are definite integrals, which are linear functionals, and indefinite integrals, which are linear operators. The further categorization of definite integrals constructed in [M. Burgin, “Directions and methods in integration”, in: Functional analysis and probability. New York, NY: Nova Science Publishers. 127-146 (2015)] contains three basic types of integration methodologies: constructive approaches and techniques, axiomatic approaches and techniques, and mixed approaches and techniques. An interesting and important problem in the area of integration is to find relations between constructive and axiomatic types of integrals. Namely, it is possible to ask two questions: what classes of linear functionals, which form the basis for axiomatic approaches, can be represented as constructive integrals, and how to axiomatically characterize constructive integrals in the class of all linear functionals.

The paper under review makes a substantial advancement in the second direction, which originated with Jacques Hadamard and Maurice Fréchet. In his paper [C. R. Acad. Sci., Paris 136, 351-354 (1903; JFM 34.0419.06)], J. Hadamard proved that it is possible to represent every linear bounded functional \( U(f) \) on the space \( C[a,b] \) of all continuous functions in the interval \([a,b]\) as a limit of the Cauchy-Riemann integrals where

\[
U(f) = \lim_{n \to \infty} \int_a^b f(x)H_n(x)dx
\]

In his turn, M. Fréchet proved [Trans. Am. Math. Soc. 5, 493-499 (1904; JFM 35.0389.03)] that any bounded linear functional \( U(f) \) on the space \( C^\infty[a,b] \) can be represented in the form

\[
U(f) = \sum_{i=0}^{n-1} A_i f^{(i)}(x) + \lim_{p \to x} \int_a^b f^{(n)}(x)H_p(x)dx.
\]

The first faithful representation of linear functionals by constructive integrals was obtained by F. Riesz [C. R. Acad. Sci., Paris 149, 974-977 (1910; JFM 40.0388.03)], who proved that the space of all Riemann-Stieltjes integrals on \( C[a,b] \) coincides with the space of all linear bounded functionals on \( C[a,b] \). This was also an axiomatic representation of Riemann-Stieltjes integrals.

Later many mathematicians, such as J. Radon, F. Hausdorff, S. Banach, S. Saks, S. Kakutani, P. Halmos, E. Hewitt, R. E. Edwards, Yu. V. Prokhorov, N. Bourbaki, and H. König, contributed to this direction. In a series of papers, the authors continued these investigations giving new representations of linear functionals by constructive integrals. The main results of the paper under review are:

Theorem A, where it is proved that the cone of all integrals on an arbitrary Hausdorff space \( T \) with respect to positive Radon measures on \( T \) coincides with the cone of all positive exact linear functionals on the truncated lattice linear subspace \( A(T) \) of universally integrable functions on the space \( S(T) \) of symmetrically functions if \( A(T) \) satisfies additional conditions;

Theorem B, where under the conditions of Theorem A, it is proved that taking Radon bimeasures on \( T \) instead of Radon measures, it is possible to prove similar results without positivity;

and Theorem C, where characteristics of the lattice linear space of all Radon bimeasures on \( T \) are obtained.

Linear functionals and definite integrals have essential limitations. For instance, such simple integrals as \( \int_{-\infty}^{\infty} dx \) or \( \int_0^\infty xdx \) cannot be properly defined. To overcome these and other limitations related to integration in infinite dimensional spaces, the theory of hyperintegration and linear hyperfunctionals has been created (cf., for example, [M. Burgin, Hypernumbers and extrafunctions. Extending the classical...]}
calculus. New York, NY: Springer (2012; Zbl 1253.46050)]. Thus, it would be natural to study the problem of finding relations between constructive and axiomatic types of hyperintegrals answering the following questions: what classes of linear hyperfunctionals can be represented as constructive hyperintegrals, and how to axiomatically characterize constructive hyperintegrals in the class of all linear hyperfunctionals. It is necessary to remark that the paper has some minor deficiencies. For instance, the authors extensively use much elaborate abstract notation but some terms are not defined when they are used the first time. Besides, some statements related to the history of integration are not correct. For instance, the authors claim that Hadamard [loc. cit.] and Fréchet [loc. cit.] “solved the following natural problem: to characterize the Riemann-Stieltjes integrals among all linear functionals on $C[a, b]$.” Actually only Riesz did this in 1909.

Reviewer: Mark S. Burgin (Los Angeles)

MSC:
28C05 Integration theory via linear functionals (Radon measures, Daniell integrals, etc.), representing set functions and measures

Full Text: DOI

References: