Zelditch, Steve
Park city lectures on eigenfunctions. (English) [Zbl 1365.58017]
Bray, Hubert L. (ed.) et al., Geometric analysis. Lecture notes from the graduate minicourse of the
2013 IAS/Park City Mathematics Institute session on geometric analysis, Park City, UT, USA, 2013.
Providence, RI: American Mathematical Society (AMS); Princeton, NJ: Institute for Advanced Study
111-193 (2016).

The paper under review consists in notes from a series of lectures on the geometry of eigenfunctions
on Riemannian manifolds. More precisely, let \((M, g)\) be a (complete) Riemannian manifold of dimension
\(m\), equipped with a Riemannian metric \(g\). Denote by \(C_0(M)\) the space of compactly supported smooth
functions on \(M\) and by \(L^2(M)\) the space of square integrable functions on \(M\) with respect to the volume
form induced by the metric \(g\). The corresponding (negative) Laplacian \(\Delta_g\) may be viewed as an unbounded
essentially self-adjoint operator on \(C_0^\infty(M) \subset L^2(M)\) induced by the Dirichlet form. The main focus of
the lectures is the study of the eigenfunctions

\[ \Delta_g \varphi_\lambda = \varphi_\lambda \]

in terms of the geometry of the nodal hypersurfaces

\[ N_{\varphi_\lambda} := \{ x \in M \mid \varphi_\lambda(x) = 0 \} \]

and their volumes \(H^{m-1}(N_{\varphi_\lambda})\). Among the numerous results stated in the notes, one can mention
Proposition 1: For any \((M, g)\) there exists a constant \(A > 0\) so that every ball of \((M, g)\) of radius greater
than \(\frac{A}{\lambda}\) contains a nodal point of any eigenfunction \(\varphi_\lambda\).

Theorem 2: 

\[ \lambda^{1-\frac{m-1}{2}} \leq CH^{m-1}(N_{\varphi_\lambda}) \]

for some constant \(C\) independent of \(\lambda\). Along with results on the number of intersections of nodal sets
with geodesics; the dynamics of the billiard flow; the \(L^p\) norms and growth properties of eigenfunctions
or the quasi-modes.

The text under review contains a tremendous amount of information with clear statements, instructive
proofs, enlightening examples and a large bibliography. These notes provide an extensive and detailed
survey on spectral geometry of Riemannian manifolds for both experts and advanced graduate students.

For the entire collection see [Zbl 1343.53002].

Reviewer: Salah Mehdi (Metz)

MSC:

58J50 Spectral problems; spectral geometry; scattering theory on manifolds
53C21 Methods of global Riemannian geometry, including PDE methods; curvature restrictions
53-02 Research exposition (monographs, survey articles) pertaining to differential geometry

Keywords:
eigenfunction; Laplacian; Riemannian manifold; nodal geometry; semi-classical analysis; quasi-mode; dynamics; billiard flow

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