In Parts I and II [Publ. Math., Inst. Hautes Étud. Sci. 37, 27–115 (1969; Zbl 0284.14008); Ann. Math. (2) 80, 227–299 (1964; Zbl 1367.14006)] the author showed that the zeta-function of a non-singular hypersurface defined over GF($q$) has the overall form predicted by the Weil conjectures and satisfies the right functional equation. The present paper breaks fresh ground by considering the singular hypersurfaces; except for the rationality, little is known (or even conjectured) about the zeta-function of these.

The notations, ideas, and to some extent the methods of the previous papers are assumed as a prerequisite. In the first two papers, finite-dimensional quotient spaces of $p$-adic power series were introduced which served as $p$-adic cohomology groups for the non-singular hypersurface. (An explicit isomorphism with the deRham cohomology has since been constructed by [N. M. Katz, Publ. Math., Inst. Hautes Étud. Sci. 35, 71–106 (1968; Zbl 0159.22502)].) In this paper, analogous cohomology spaces $H^s(L^*)$ are introduced for the singular hypersurface, and the bulk of the paper is devoted to proving they are finite-dimensional. These spaces $H^s(L^*)$ are constructed, as before, from a space $L^*$ of $p$-adic Laurent series in several variables, the cohomology being defined via the differential operators $D_i$ derived from the polynomial $f$ defining the hypersurface. The proof that the spaces $H^s(L^*)$ are finite-dimensional is by induction on $s$, the space $H^s(L^*)$ being related to the cohomology space $H^{s-1}(L^*)$ of a higher-dimensional hypersurface – essentially a one-parameter family $H$ of hypersurfaces, all non-singular except for the given hypersurface $H_0$. (Thus, the deformation theory given in Part II plays an essential role here, too.) The finite-dimensionality of the space $H^1(L^*)$ comes from results in $p$-adic ordinary differential equations, to which a brief chapter is devoted.

A second type of cohomology space $\hat{K}_\infty \subset L^*$ is introduced on which the basic endomorphism $\alpha^*$ acts, and which is spanned by the eigenvectors of $\alpha^*$. (In the non-singular case, the characteristic polynomial of $\alpha^*$ is essentially the zeta-function.) Its cohomology is also proved to be finite-dimensional, provided $f$ has coefficients in an algebraic number field. The representation of $\alpha^*$ on the spaces $H^s(\hat{K}_\infty)$ leads to a decomposition of $\det(I - \lambda\alpha^*)$ into the characteristic polynomials of the endomorphism acting on these spaces.

The relation of these cohomology spaces to classical ones is left open, nor are any explicit conjectures about the zeta-function offered, for singular hypersurfaces. In this connection, however, the author remarks that the mapping between the cohomology space and its dual which gave the functional equation for non-singular hypersurfaces still exists in this new theory, and has the same formal relation with $\alpha^*$; however, it is no longer an isomorphism. This as well as other clues to the future direction of the theory are offered at the end of the introduction.

Reviewer: A. Mattuck (MR 35,194)

MSC: 14G10 Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture) 14G15 Finite ground fields in algebraic geometry 14J20 Arithmetic ground fields for surfaces or higher-dimensional varieties

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