Let $M, N$ be two bracket-generating sub-Riemannian manifolds, and suppose that $F : M \to N$ is a (surjective) isometry of the control (Carnot-Carathéodory) distances. Must $F$ be smooth with respect to the differentiable structures of the underlying manifolds of $M, N$, and hence be a diffeomorphism? The analogous statement for Riemannian manifolds is well known to be true, but the sub-Riemannian case is more difficult since, unlike the Riemannian distance, the control distance need not be smooth.

In this paper, the authors give a positive answer under the assumption that $M, N$ are equiregular. Let $\Delta^1 = \Delta$ denote the horizontal distribution of $M$, and let $\Delta^{i+1} = [\Delta, \Delta^i]$. We say $\Delta$ is equiregular on an open set $U$ if the dimension of $\Delta^i_p$ is independent of $p \in U$. As an immediate generalization of the result, it follows that an isometry $F : M \to N$ is smooth on any open set $U$ on which the distribution of $M$ is locally equiregular; in particular, there exists such $U$ which is dense.

The proof proceeds in two steps which are of independent interest. First, the desired result is shown for general sub-Riemannian manifolds (not necessarily equiregular) under the additional hypothesis that there exist smooth volume forms $\text{vol}_M, \text{vol}_N$ on $M, N$ such that the isometry $F$ pushes forward $\text{vol}_M$ to $\text{vol}_N$, i.e., $F_*\text{vol}_M = \text{vol}_N$. The smoothness of $F$ can then be shown by a bootstrap argument based on subelliptic estimates for the sub-Laplacian. Second, it is shown that for equiregular sub-Riemannian manifolds, the canonical Popp volume forms satisfy this hypothesis; this is proved by relating the Popp measure to the spherical Hausdorff measure, where the latter depends only on the metric structure.

As a consequence, taking $N = M$, the authors show that if $M$ is an equiregular sub-Riemannian manifold, then the isometry group of $M$ is a finite-dimensional Lie group; moreover, for each compact subgroup $K$, there is a Riemannian extension $g_K$ of the sub-Riemannian metric of $M$ such that $K$ embeds in the isometry group of the Riemannian manifold $(M, g_K)$. This holds in particular when $K$ is the group of isometries fixing a particular point $p \in M$.

Another corollary is that if $M, N$ are equiregular and connected and a point $p \in M$ is fixed, then any isometry $F : M \to N$ is uniquely determined by the value of $F(p)$ and the action of the differential $dF$ on the horizontal space $\Delta_p$ at $p$.

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