Ramos, Vinicius Gripp Barros
Symplectic embeddings and the Lagrangian bidisk. (English) Zbl 1370.53057
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In the study of symplectic manifolds symplectic embedding problems play a significant role. Many techniques were created to deal with questions about when symplectic embeddings exist. Symplectic capacities are one such technique, and they provide an obstruction to the existence of a symplectic embedding. If for a symplectic four-dimensional manifold (X,ω), there is a sequence of real numbers $c_k(X,\omega)$ such that

$$0 = c_0(X,\omega) < c_1(X,\omega) \le c_2(X,\omega) \le \cdots \le \infty,$$

then they are called embedded contact homology (ECH) capacities. The capacities satisfy the following properties:

(i) if $a > 0$, then $c_k(X, a\cdot\omega) = a\cdot c_k(X,\omega)$ for every k,

(ii) if $(X_1,\omega_1) \hookrightarrow (X_2,\omega_2)$, then $c_k(X_1,\omega_1) \le c_k(X_2,\omega_2)$ for all k, and

(iii) $c_k\left(\coprod_{i=1}^{n}(X_i,\omega_i)\right) = \max\left\{\sum_{i=1}^{n} c_{k_i}(X_i,\omega_i);\ k_1+\cdots+k_n = k\right\}$.

For a certain embedding problem $(X_1,\omega_1) \hookrightarrow (X_2,\omega_2)$ ECH capacities are said to be sharp if $\forall_k\ c_k(X_1,\omega_1) \le c_k(X_2,\omega_2) \implies (X_1,\omega_1) \hookrightarrow (X_2,\omega_2)$. Any subset of $\mathbb{R}^4 \cong \mathbb{C}^2$ with coordinates $(p_1,q_1,p_2,q_2) = (z_1,z_2)$ is endowed with the symplectic form $\omega = \sum_{i=1}^{2} dp_i \wedge dq_i$. A Lagrangian bidisk P_L in $\mathbb{R}^4$ is defined as

$$P_L = \{(p_1,q_1,p_2,q_2);\ p_1^2+p_2^2 \le 1 \wedge q_1^2+q_2^2 \le 1\}$$

and the Lagrangian product of any two disks is symplectomorphic to a multiple of P_L. An ellipsoid $E(a,b)$ and a symplectic polydisk $P(a,b)$ are defined as

$$E(a,b) = \left\{(z_1,z_2);\ \pi\left(\frac{|z_1|^2}{a}+\frac{|z_2|^2}{b}\right) \le 1\right\}$$
$$P(a,b) = \{(z_1,z_2);\ \pi|z_1|^2 \le a \wedge \pi|z_2|^2 \le b\},$$

respectively. $B(a) = E(a,a)$ is the Euclidean ball of radius $\sqrt{a/\pi}$.

In this paper, the author obtains sharp obstructions to the symplectic embedding of the Lagrangian bidisk into four-dimensional balls, ellipsoids, and symplectic polydisks.

It is proven that ECH capacities give a sharp obstruction to symplectically embedding the interior of P_L into balls, ellipsoids, and symplectic polydisks. Moreover,

(i) $\operatorname{int}(P_L) \hookrightarrow B(a)$ if and only if $a \ge 3\sqrt{3}$,

(ii) $\operatorname{int}(P_L) \hookrightarrow E(a,b)$ if and only if $\min(a,b) \ge 4$ and $\max(a,b) \ge 3\sqrt{3}$, and

(iii) $\operatorname{int}(P_L) \hookrightarrow P(a,b)$ if and only if $a,b \ge 4$.

If Ω is a closed region in the first quadrant of $\mathbb{R}^2$, then the toric domain $X_\Omega \subset \mathbb{C}^2$ is defined as $X_\Omega = \{(z_1,z_2);\ \pi(|z_1|^2,|z_2|^2) \in \Omega\}$ and is endowed with the restriction of the standard symplectic form in $\mathbb{C}^2$. To prove the main result the author needs to show that if X_0 is the toric domain X_{Ω_0}, where Ω_0 is the region bounded by the coordinate axes and the curve parameterized by

$$\left(2\sin\left(\frac{\alpha}{2}\right) - \alpha\cos\left(\frac{\alpha}{2}\right), 2\sin\left(\frac{\alpha}{2}\right) + (2\pi-\alpha)\cos\left(\frac{\alpha}{2}\right)\right),\quad \alpha \in [0,2\pi],$$

then $\operatorname{int}(P_L)$ and $\operatorname{int}(X_0)$ are symplectomorphic.

Reviewer: Andrew Bucki (Edmond)

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