The André-Oort conjecture. (English) Zbl 1377.11073

This paper proves the André-Oort conjecture for arbitrary Shimura varieties, under the assumption of the generalized Riemann hypothesis for CM fields or a technical hypothesis on the set of special points in question. Recall that a Shimura variety $\text{Sh}(G,X)$ is an inverse system of smooth quasi-projective projective varieties over $\mathbb{C}$ attached to a connected reductive group $G$ over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class of homomorphisms $X \subset \text{Hom}(\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, G_{\mathbb{R}})$ satisfying certain axioms (a Shimura datum), indexed by sufficiently small compact open subgroups $K \subset G(\mathbb{A})$. The formation of $\text{Sh}(G,X)$ is functorial with respect to an obvious notion of morphism of Shimura data. The group $G(\mathbb{A})$ also acts on the right on the tower $\text{Sh}(G,X)$ by algebraic correspondences, called Hecke correspondences.

In the context of Shimura varieties, one may define a subvariety $V \subset \text{Sh}_K(G,X_G)$ to be special if there exists a Shimura datum $(H,X_H)$, a morphism $(H,X_H) \rightarrow (G,X_G)$, and an element $g \in G(\mathbb{A})$ such that $V$ is an irreducible component of the image of the composite

$$\text{Sh}(H,X_H) \rightarrow \text{Sh}(G,X_G) \cdot g \rightarrow \text{Sh}(G,X_G) \text{Sh}_K(G,X_G).$$

(As is pointed out in the paper’s introduction, the notion of special can also be understood more generally, in abstract Hodge-theoretic terms.) A special point is a special subvariety of dimension zero.

If $V \subset \text{Sh}_K(G,X)$ is special, then one can show that the special points in $\text{Sh}_K(G,X)(\mathbb{C})$ contained in $V$ form a dense subset of $V$ for the strong (and hence for the Zariski) topology. The André-Oort conjecture asserts the converse: that every irreducible component of the Zariski-closure of a collection $\Sigma \subset \text{Sh}_K(G,X)(\mathbb{C})$ of special points is a special subvariety.

The main result of the paper under review is to prove the André-Oort conjecture under the assumption of the generalized Riemann hypothesis for CM fields, or under a technical hypothesis on the Mumford-Tate groups of the points in $\Sigma$. In fact, the authors prove a natural generalization where $\Sigma$ may be any collection of special subvarieties, not just points (subject to analogous assumptions).

The proof is rooted in the strategy of B. Edixhoven and the second author [Ann. Math. (2) 157, No. 2, 621–645 (2003; Zbl 1053.14023)], who proved the conjecture for curves in Shimura varieties containing infinite sets of special points satisfying the technical hypothesis alluded to above. Two main difficulties arise here. One is the question of irreducibility of transforms of subvarieties under Hecke correspondences. The other is the issue of higher dimensional special subvarieties, which the authors deal with via an inductive technique based on previous work of E. Ullmo and the second author [Ann. Math. (2) 180, No. 3, 823–865 (2014; Zbl 1328.11070)].

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